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Multivariate Objective Analysis of Temperature and Wind Fields

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I. Introduction

This paper describes the design of an objective analysis system based upon the statistical optimum interpolation principles of Gandin (1963). The proposed analysis system interpolates values of meteorological variables to grid point locations. The analysis system is multivariate in that horizontal wind observations are used in the analysis of the temperature field and vice versa. The thermal wind relationship is used as a weak constraint. Moreover, the analysis scheme is three-dimensional. Analysis may proceed directly to model sigma surfaces (or any other desired analysis output surface) from observations distributed in three dimensions. Finally, the analysis scheme will make allowance for the various error levels of the heterogeneous observational data base in a rational and systematic way in producing the final analysis.

Multivariate objective analysis systems for the analysis of geopotential height and horizontal wind fields have been developed and tested by Kluge (1970), Rutherford (1973), and Schlatter (1975). In these systems, the geostrophic relationship between heights and winds is used as a weak constraint in performing the multivariate analyses. "First-guess" fields are provided by forecast values, deviations of observations from forecasts are computed, and these deviational values are then analyzed by the multivariate optimum interpolation method. The geostrophic assumption is invoked to relate the covariances of wind-component and height forecast errors. Buell (1958, 1972) has shown that the covariances of wind components and heights are indeed related in an approximately geostrophic manner at middle and high latitudes. The conclusion of all three investigations is that the height analysis is significantly improved by multivariate analysis including wind data as compared to a univariate analysis using height information only. On the other hand, Rutherford and Schlatter conclude that use of height information in a multivariate wind analysis does not significantly improve the wind analysis.

The analysis system proposed here is similar to those of Rutherford and Schlatter except that temperature replaces geopotential height as the mass variable to be analyzed multivariately with the wind components, and the three-dimensional thermal wind relation replaces the geostrophic relation between the covariances of the analyzed variables. Although it would be possible to use cross covariances of the variables determined from actual data samples such as those determined by Buell, use of a dynamic relationship has two advantages: (1) The dynamic constraint will provide a better balancing of the mass and momentum fields as an initial state for model integration, and (2) the covariances can be expressed as relatively simple analytic functions rather than needing to be stored in the form of very large tables for computer access. Moreover, the experience of other optimum interpolation analysts (Gandin, 1963; Schlatter, 1975) is that the precise forms of the covariance functions are not as important as their general characteristics, and that analytic approximations are quite suitable for the purpose of producing a statistically optimum analysis from the available data.

In equatorial latitudes, however, the use of the thermal wind relation as even a weak constraint is not appropriate. Therefore, the analysis scheme will be designed to gradually decouple the temperature-wind interdependence as the equator is approached, so that the analyses of temperature and wind fields will be essentially independent of each other in the immediate vicinity of the equator. The degree of decoupling as a function of latitude will be determined on the basis of experimentation.

The results of Rutherford and Schlatter mentioned above suggest that inclusion of temperature information in the wind analyses may not significantly improve the results over those using wind information only. However, their conclusions assume the presence of equal amounts of temperature and wind data. It is anticipated that for large portions of the globe, density of temperature data will greatly exceed density of wind data; therefore, in these regions at least, multivariate use of temperature data in the wind analyses is indicated.

II. Theory

Let $$T = \tilde{T} + t, \quad U = \tilde{U} + u, \quad V = \tilde{V} + v, \tag{1}$$

where T, U, and V are the total values of temperature, westerly, and southerly wind component of either an observation or of the analyzed value at a grid point. The quantities $\tilde{T}$, $\tilde{U}$, and $\tilde{V}$ are "guess" values (provided by a forecast, climatology, or a suitable blend of the two) at observation or grid points, and t, u, v are deviations of the total values from the "guess" values. Then the linear multivariate analysis equation for the temperature T_g at a grid point is given by

$$T_g \approx \tilde{T}_g + \sum_{i=1}^{\ell} a_i t_i + \sum_{j=1}^{m} b_j u_j + \sum_{k=1}^{n} c_k v_k \, , \tag{2}$$

where $\tilde{T}_g$ is the "guess" temperature at the grid point, and a_i, b_j, c_k are weighting factors for the ℓ temperature observations, m u-component observations, and n v-component observations within a specified volume about the grid point. This volume is chosen sufficiently large so as to include all observations having a significant weight in determining the analyzed value at the grid point. In the absence of any such observations, the "guess" value becomes the analyzed value. Similar multivariate expressions may be written for U_g and V_g .

In general, the right-hand-side of (2) will not give the true value of T_g at the grid point but will differ by an amount called the analysis error. Since the actual true value T_g in any given situation is unknown, it is not possible to eliminate this source of error exactly, no matter how the weighting factors are chosen. We can, however, stipulate that the mean* square error of equation (2)

$$E = \overline{\left(T_g - \tilde{T}_g - \sum_{i=1}^{\ell} a_i t_i - \sum_{j=1}^{m} b_j u_j - \sum_{k=1}^{n} c_k v_k \right)^2} \tag{3}$$

be a minimum and use this requirement to determine the values of the weighting factors.

We now require that

$$\begin{aligned} \partial E / \partial a_i &= 0 \, , & i &= 1, \ell \\ \partial E / \partial b_j &= 0 \, , & j &= 1, m \\ \partial E / \partial c_k &= 0 \, , & k &= 1, n \end{aligned} \tag{4}$$

The result is the following set of equations:

$$\sum_{i=1}^{\ell} \overline{t_{\ell'} t_i}\, a_i + \sum_{j=1}^{m} \overline{t_{\ell'} u_j}\, b_j + \sum_{k=1}^{n} \overline{t_{\ell'} v_k}\, c_k = \overline{t_{\ell'} t_g} \, , \quad \ell' = 1, \ell \tag{5a}$$

*The mean to be taken over a large sample of observational data.

$$\sum_{i=1}^{\ell} \overline{u_{m'} t_i}\, a_i + \sum_{j=1}^{m} \overline{u_{m'} u_j}\, b_j + \sum_{k=1}^{n} \overline{u_{m'} v_n}\, c_k = \overline{u_{m'} t_g}, \quad m' = 1, m \tag{5b}$$

$$\sum_{i=1}^{\ell} \overline{v_{n'} t_i}\, a_i + \sum_{j=1}^{m} \overline{v_{n'} u_j}\, b_j + \sum_{k=1}^{n} \overline{v_{n'} v_k}\, c_k = \overline{v_{n'} t_g}, \quad n' = 1, n \tag{5c}$$

The equations resulting from minimizing analysis error associated with U_g or V_g are the same except that, in the right-hand-side term, t_g is replaced by u_g or v_g, respectively. (Of course, the resulting solutions for the weighting coefficients a_i, b_j, and c_k will be different in each case.)

The quantities $\overline{t_{\ell'} t_i}$, $\overline{t_{\ell'} u_j}$, etc., are <u>covariances</u> of the deviational quantities t, u, v. The values of these covariances will, in actuality, be "inflated" by the presence of observational errors. For now, we will consider only the "true" component of the covariances. It is a relatively easy matter to add on the error terms, as will be shown in a later section.

It is convenient to express equations (5) in normalized form. To do this, divide (5a) by $\sqrt{\overline{t_{\ell'}^2}\;\overline{t_g^2}}$, (5b) by $\sqrt{\overline{u_{m'}^2}\;\overline{t_g^2}}$, and (5c) by $\sqrt{\overline{v_{n'}^2}\;\overline{t_g^2}}$ to get

$$\sum_{i=1}^{\ell} \rho_{\ell' i}^{tt} a_i' + \sum_{j=1}^{m} \rho_{\ell' j}^{tu} b_j' + \sum_{k=1}^{n} \rho_{\ell' k}^{tv} c_k' = \rho_{\ell' g}^{tt}, \quad \ell' = 1, \ell \tag{6a}$$

$$\sum_{i=1}^{\ell} \rho_{m' i}^{ut} a_i' + \sum_{j=1}^{m} \rho_{m' j}^{uu} b_j' + \sum_{k=1}^{n} \rho_{m' k}^{uv} c_k' = \rho_{m' g}^{ut}, \quad m' = 1, m \tag{6b}$$

$$\sum_{i=1}^{\ell} \rho_{n' i}^{vt} a_i' + \sum_{j=1}^{m} \rho_{n' j}^{vu} b_j' + \sum_{k=1}^{n} \rho_{n' k}^{vv} c_k' = \rho_{n' g}^{vt}, \quad n' = 1, n \tag{6c}$$

where

$$\rho^{tt}_{l'i} \equiv \frac{\overline{t_{l'} t_i}}{\sqrt{\overline{t^2_{l'}}\ \overline{t^2_i}}}, \qquad \rho^{tu}_{l'j} \equiv \frac{\overline{t_{l'} u_j}}{\sqrt{\overline{t^2_{l'}}\ \overline{u^2_j}}}, \quad etc., \tag{7}$$

and

$$a_i' \equiv \sqrt{\frac{\overline{t^2_i}}{\overline{t^2_g}}}\, a_i, \quad b_j' \equiv \sqrt{\frac{\overline{u^2_j}}{\overline{t^2_g}}}, \quad c_k' \equiv \sqrt{\frac{\overline{v^2_k}}{\overline{t^2_g}}}\, c_k. \tag{8}$$

We will return to these normalized forms in terms of correlation functions ρ, rather than covariances, later.

We now proceed to evaluate the individual covariances. Assume that the field of temperature deviations (t) is homogeneous within the volume about a grid point, and that the temperature covariance can be approximated by a simple analytic expression. Assume additionally, that the other eight covariances of (5) are related to $\overline{t_i t_j}$ through the thermal wind relations

$$u^* \equiv \frac{\partial u}{\partial p} = \frac{R}{fp}\frac{\partial t}{\partial y} \tag{9a}$$

$$v^* \equiv \frac{\partial v}{\partial p} = -\frac{R}{fp}\frac{\partial t}{\partial x} \tag{9b}$$

Then the corresponding covariances are related to $\overline{t_i t_j}$ by

$$\overline{u_i^* t_j} = \frac{R}{f_i p_i}\frac{\partial}{\partial y_i}(\overline{t_i t_j}) \tag{10a}$$

$$\overline{t_i u_j^*} = \frac{R}{f_j \rho_j} \frac{\partial}{\partial y_j} (\overline{t_i t_j}) \tag{10b}$$

$$\overline{v_i^* t_j} = -\frac{R}{f_i \rho_i} \frac{\partial}{\partial x_i} (\overline{t_i t_j}) \tag{10c}$$

$$\overline{t_i v_j^*} = -\frac{R}{f_j \rho_j} \frac{\partial}{\partial x_j} (\overline{t_i t_j}) \tag{10d}$$

$$\overline{u_i^* u_j^*} = \frac{R}{f_i \rho_i} \frac{\partial}{\partial y_i} (\overline{t_i u_j^*}) = \frac{R^2}{f_i f_j \rho_i \rho_j} \frac{\partial^2}{\partial y_i \partial y_j} (\overline{t_i t_j}) \tag{10e}$$

$$\overline{v_i^* v_j^*} = -\frac{R}{f_i \rho_i} \frac{\partial}{\partial x_i} (\overline{t_i v_j^*}) = \frac{R^2}{f_i f_j \rho_i \rho_j} \frac{\partial^2}{\partial x_i \partial x_j} (\overline{t_i t_j}) \tag{10f}$$

$$\overline{u_i^* v_j^*} = \frac{R}{f_i \rho_i} \frac{\partial}{\partial y_i} (\overline{t_i v_j^*}) = -\frac{R^2}{f_i f_j \rho_i \rho_j} \frac{\partial^2}{\partial y_i \partial x_j} (\overline{t_i t_j}) \tag{10g}$$

$$\overline{v_i^* u_j^*} = -\frac{R}{f_i \rho_i} \frac{\partial}{\partial x_i} (\overline{t_i u_j^*}) = -\frac{R^2}{f_i f_j \rho_i \rho_j} \frac{\partial^2}{\partial x_i \partial y_j} (\overline{t_i t_j}) \tag{10h}$$

In the above, derivatives of f with respect to y have been neglected. (See Appendix for proofs of the above derivative relations.)

Additionally, the following covariance relations may be obtained from (9a, b):

$$\overline{u_i^* t_j} = \frac{\partial}{\partial p_i}(\overline{u_i t_j}), \qquad \overline{t_i u_j^*} = \frac{\partial}{\partial p_j}(\overline{t_i u_j}), \tag{11a, b}$$

$$\overline{v_i^* t_j} = \frac{\partial}{\partial p_i}(\overline{v_i t_j}), \qquad \overline{t_i v_j^*} = \frac{\partial}{\partial p_j}(\overline{t_i v_j}), \tag{11c, d}$$

$$\overline{u_i^* u_j^*} = \frac{\partial^2}{\partial p_i \partial p_j}(\overline{u_i u_j}), \qquad \overline{v_i^* v_j^*} = \frac{\partial^2}{\partial p_i \partial p_j}(\overline{v_i v_j}) \tag{11e, f}$$

$$\overline{u_i^* v_j^*} = \frac{\partial^2}{\partial p_i \partial p_j}(\overline{u_i v_j}), \qquad \overline{v_i^* u_j^*} = \frac{\partial^2}{\partial p_i \partial p_j}(\overline{v_i u_j}). \tag{11g, h}$$

Integrating equations (11) and using equations (10),

$$\overline{u_i t_j} = \overline{u_{io} t_j} + \int_{p_{io}}^{p_i} \overline{u_i^* t_j}\, dp_i = \frac{R}{f_i} \int\!\!\int_{p_{io}}^{p_i} \left[\frac{\partial}{\partial y_i}(\overline{t_i t_j})\right] \frac{dp_i}{p_i} \tag{12a}$$

$$\overline{t_i u_j} = \overline{t_i u_{jo}} + \int_{p_{jo}}^{p_j} \overline{t_i u_j^*}\, dp_j = \frac{R}{f_j} \int_{p_{jo}}^{p_j} \left[\frac{\partial}{\partial y_j}\left(\overline{t_i t_j}\right)\right] \frac{dp_j}{p_j} \tag{12b}$$

$$\overline{v_i t_j} = \overline{v_{io} t_j} + \int_{p_{io}}^{p_i} \overline{v_i^* t_j}\, dp_i = -\frac{R}{f_i} \int_{p_{io}}^{p_i} \left[\frac{\partial}{\partial x_i}\left(\overline{t_i t_j}\right)\right] \frac{dp_i}{p_i} \tag{12c}$$

$$\overline{t_i v_j} = \overline{t_i v_{jo}} + \int_{p_{jo}}^{p_j} \overline{t_i v_j^*}\, dp_j = -\frac{R}{f_j} \int_{p_{jo}}^{p_j} \left[\frac{\partial}{\partial x_j}\left(\overline{t_i t_j}\right)\right] \frac{dp_j}{p_j} \tag{12d}$$

$$\overline{u_i u_j} = \overline{u_{io} u_{jo}} + \int_{p_{jo}}^{p_j}\int_{p_{io}}^{p_i} \overline{u_i^* u_j^*}\, dp_i\, dp_j = \frac{R^2}{f_i f_j} \int_{p_{jo}}^{p_j}\int_{p_{io}}^{p_i} \left[\frac{\partial^2}{\partial y_i \partial y_j}\left(\overline{t_i t_j}\right)\right] \frac{dp_i}{p_i}\frac{dp_j}{p_j} \tag{12e}$$

$$\overline{v_i v_j} = \overline{v_{io} v_{jo}} + \int_{p_{jo}}^{p_j}\int_{p_{io}}^{p_i} \overline{v_i^* v_j^*}\, dp_i\, dp_j = \frac{R^2}{f_i f_j} \int_{p_{jo}}^{p_j}\int_{p_{io}}^{p_i} \left[\frac{\partial^2}{\partial x_i \partial x_j}\left(\overline{t_i t_j}\right)\right] \frac{dp_i}{p_i}\frac{dp_j}{p_j} \tag{12f}$$

$$\overline{u_i v_j} = \overline{u_{io} v_{jo}} + \int_{p_{jo}}^{p_j}\int_{p_{io}}^{p_i} \overline{u_i^* v_j^*}\, dp_i\, dp_j = \frac{-R^2}{f_i f_j} \int_{p_{jo}}^{p_j}\int_{p_{io}}^{p_i} \left[\frac{\partial^2}{\partial y_i \partial x_j}\left(\overline{t_i t_j}\right)\right] \frac{dp_i}{p_i}\frac{dp_j}{p_j} \tag{12g}$$

$$\overline{v_i u_j} = \overline{v_{io} u_{jo}} + \int_{p_{jo}}^{p_j}\int_{p_{io}}^{p_i} \overline{v_i^* u_j^*}\, dp_i\, dp_j = \frac{-R^2}{f_i f_j}\int_{p_{jo}}^{p_j}\int_{p_{io}}^{p_i}\left[\frac{\partial^2}{\partial x_i \partial y_j}(\overline{t_i t_j})\right]\frac{dp_i}{p_i}\frac{dp_j}{p_j} \qquad (12h)$$

where p_{io}, p_{jo} refer to the surface pressure corresponding to the (i)th and (j)th observation locations. The covariances $\overline{u_{io} t_j}$, etc., are assumed to be negligibly small and are neglected on the right-hand sides of (12a-d). The covariances $\overline{u_{io} u_{jo}}$, etc., have also been neglected on the right-hand sides of (12e-h) but with less justification. These covariances are neglected in the following because (1) they are difficult to incorporate into the theoretical development, and (2) the results of the theory strongly suggest that the terms $\overline{u_{io} u_{jo}}$, etc., must indeed be small compared to the integral terms of (12e-h). We will return to a discussion of this problem after examining the results.

We will now assume that the deviational temperature covariance $\overline{t_i t_j}$ can be approximated by an analytic expression of the form

$$\overline{t_i t_j}(x_i, y_i, p_i ; x_j, y_j, p_j) \approx C\,\psi(x_i, y_i ; x_j, y_j) \cdot \chi(p_i, p_j) \qquad (13)$$

where C is the deviational temperature variance, assumed constant within the influence volume for the analysis at a particular grid point, but may vary as the analysis system moves from one grid point to another. In (13), the dependence of the covariance on the horizontal separation of a pair of observations (or of an observation and the analysis point) is assumed to be independent of their vertical separation and vice versa. Substitution of (13) in (12a-h) leads to

$$\overline{u_i t_j} = \frac{CR}{f_i}\frac{\partial \psi}{\partial y_i}\int_{p_{io}}^{p_i} \chi \frac{dp_i}{p_i}, \qquad \overline{t_i u_j} = \frac{CR}{f_j}\frac{\partial \psi}{\partial y_j}\int_{p_{jo}}^{p_j} \chi \frac{dp_j}{p_j}, \qquad (14a, b)$$

$$\overline{v_i t_j} = -\frac{CR}{f_i}\frac{\partial\psi}{\partial x_i}\int_{p_{io}}^{p_i}\chi\frac{dp_i}{p_i}, \qquad \overline{t_i v_j} = -\frac{CR}{f_j}\frac{\partial\psi}{\partial x_j}\int_{p_{jo}}^{p_j}\chi\frac{dp_j}{p_j}, \tag{14c, d}$$

$$\overline{u_i u_j} = \frac{CR^2}{f_i f_j}\frac{\partial^2\psi}{\partial y_i \partial y_j}\int_{p_{jo}}^{p_j}\int_{p_{io}}^{p_i}\left(\chi\frac{dp_i}{p_i}\right)\frac{dp_j}{p_j} \tag{14e}$$

$$\overline{v_i v_j} = \frac{CR^2}{f_i f_j}\frac{\partial^2\psi}{\partial x_i \partial x_j}\int_{p_{jo}}^{p_j}\int_{p_{io}}^{p_i}\left(\chi\frac{dp_i}{p_i}\right)\frac{dp_j}{p_j} \tag{14f}$$

$$\overline{u_i v_j} = -\frac{CR^2}{f_i f_j}\frac{\partial^2\psi}{\partial y_i \partial x_j}\int_{p_{jo}}^{p_j}\int_{p_{io}}^{p_i}\left(\chi\frac{dp_i}{p_i}\right)\frac{dp_j}{p_j} \tag{14g}$$

$$\overline{v_i u_j} = -\frac{CR^2}{f_i f_j}\frac{\partial^2\psi}{\partial x_i \partial y_j}\int_{p_{jo}}^{p_j}\int_{p_{io}}^{p_i}\left(\chi\frac{dp_i}{p_i}\right)\frac{dp_j}{p_j} \tag{14h}$$

We will deal first with the function $\chi(p_i, p_j)$ and its integrals. The original intention was to use the form

$$\chi(p_i, p_j) = \exp\left[-k_p\left(\ln p_i - \ln p_j\right)^2\right] = \exp\left[-k_p \ln^2\left(p_i/p_j\right)\right], \tag{15a}$$

which assumes a normal, or "Gaussian," distribution for the deviational temperature covariance as a function of $\ln p$, similar to the normal distribution assumed below (Eqn. 18) for the horizontal function ψ. However, the integrals appearing in (14a-h) can only be evaluated by numerical quadrature if (15a) is used. Although extensive tables of values of these integrals could be used in the computation of the derived covariances, it is more convenient to have expressions in closed analytic form for the integrals if possible. The functional form

$$\chi(p_i, p_j) = \frac{1}{1 + q_{ij}^2} \tag{15b}$$

where $$q_{ij} \equiv k_p \ln\left(p_i/p_j\right)$$

closely approximates the functional form of (22a), and its integrals have nearly the same values as those derived from (22a). The constant k_p determines the "half-width" of the covariance distribution with respect to $\ln p$; its value will have to be determined by fitting the profile (15b) to temperature covariances determined from real data.

Three different integrals appear in equation (14). They are

$$\mathcal{P}_{ij}^{i} \equiv \int_{p_{io}}^{p_i} \chi \frac{dp_i}{p_i} \tag{16a}$$

$$\mathcal{P}_{ij}^{j} \equiv \int_{\rho_{jo}}^{\rho_j} \chi \frac{d\rho_j}{\rho_j} \tag{16b}$$

$$Q_{ij} \equiv \int_{\rho_{jo}}^{\rho_j}\int_{\rho_{io}}^{\rho_i} \left(\chi \frac{d\rho_i}{\rho_i}\right) \frac{d\rho_j}{\rho_j} \tag{16c}$$

Substitution of (15b) in (16a-c) leads to

$$\mathcal{P}_{ij}^{i} = \frac{1}{k_p}\left(\tan^{-1} q_{ij} - \tan^{-1} q_{oj}\right) \tag{17a}$$

$$\mathcal{P}_{ij}^{j} = -\frac{1}{k_p}\left(\tan^{-1} q_{ij} - \tan^{-1} q_{io}\right) \tag{17b}$$

$$Q_{ij} = -\frac{1}{k_p^2}\Bigg\{ q_{ij} \tan^{-1} q_{ij} - q_{io} \tan^{-1} q_{io} - q_{oj} \tan^{-1} q_{oj} + q_{oo} \tan^{-1} q_{oo}$$

$$+ \frac{1}{2} \ln\left[\frac{(1+q_{io}^2)(1+q_{oj}^2)}{(1+q_{ij}^2)(1+q_{oo}^2)}\right]\Bigg\} \tag{17c}$$

where $\eta_{ij} \equiv k_p \ln(P_i/P_j)$, $\eta_{oj} \equiv k_p \ln(P_{io}/P_j)$,

$\eta_{io} \equiv k_p \ln(P_i/P_{jo})$, $\eta_{oo} \equiv k_p \ln(P_{io}/P_{jo})$.

Now we turn to the function ψ and its partial derivatives. In cartesian coordinates, assume

$$\psi(x_i, y_i ; x_j, y_j) \equiv \exp\left\{-k_h\left[(x_i - x_j)^2 + (y_i - y_j)^2\right]\right\} \qquad (18)$$

However, for the global model, we wish to express horizontal locations in terms of longitude λ and latitude ϕ. Rather than using the exact great-circle distance between two points (which leads to very messy expressions for the derivatives of ψ), it is more convenient to use two good approximations (to within 5 percent of the "true" distance) to the great circle distance, one valid for low and middle latitudes, the other for polar latitudes.

a. <u>Lower Latitudes</u> ($-70^{\circ}C < \phi < +70^{\circ}$)

Let $$x_i - x_j = A \cos\left(\frac{\phi_i + \phi_j}{2}\right)(\lambda_i - \lambda_j) \qquad (19a)$$

$$y_i - y_j = A(\phi_i - \phi_j) \qquad (19b)$$

where λ is longitude, ϕ is latitude, and A is the radius of the earth. This approximation is suggested by Schlatter (1975). The corresponding expression for ψ is then

$$\psi_1(\lambda_i, \phi_i ; \lambda_j, \phi_j) = \exp\left(-k_h A^2 \xi\right), \qquad (20)$$

where $$\xi \equiv \cos^2\left(\frac{\phi_i + \phi_j}{2}\right)(\lambda_i - \lambda_j)^2 + (\phi_i - \phi_j)^2 \qquad (21)$$

With the partials of (14) reexpressed in spherical coordinates, and using (16), (20), and (21),

$$\overline{u_i t_j} = \frac{CR}{Af_i} \psi_1 P_{ij}^{i} \left[-k_h A^2 \frac{\partial \xi}{\partial \phi_i} \right] \tag{22a}$$

$$\overline{t_i u_j} = \frac{CR}{Af_j} \psi_1 P_{ij}^{j} \left[-k_h A^2 \frac{\partial \xi}{\partial \phi_j} \right] \tag{22b}$$

$$\overline{v_i t_j} = \frac{-CR}{Af_i \cos\phi_i} \psi_1 P_{ij}^{i} \left[-k_h A^2 \frac{\partial \xi}{\partial \lambda_i} \right] \tag{22c}$$

$$\overline{t_i v_j} = \frac{-CR}{Af_j \cos\phi_j} \psi_1 P_{ij}^{j} \left[-k_h A^2 \frac{\partial \xi}{\partial \lambda_j} \right] \tag{22d}$$

$$\overline{u_i u_j} = \frac{CR^2}{A^2 f_i f_j} \psi_1 Q_{ij} \left[-k_h A^2 \frac{\partial^2 \xi}{\partial \phi_i \partial \phi_j} + k_h^2 A^4 \frac{\partial \xi}{\partial \phi_i} \frac{\partial \xi}{\partial \phi_j} \right] \tag{22e}$$

$$\overline{v_i v_j} = \frac{CR^2}{A^2 f_i f_j \cos\phi_i \cos\phi_j} \psi_1 Q_{ij} \left[-k_h A^2 \frac{\partial^2 \xi}{\partial \lambda_i \partial \lambda_j} + k_h^2 A^4 \frac{\partial \xi}{\partial \lambda_i} \frac{\partial \xi}{\partial \lambda_j} \right] \tag{22f}$$

$$\overline{u_i v_j} = \frac{-CR^2}{A^2 f_i f_j \cos\phi_j} \psi_1 Q_{ij} \left[-k_h A^2 \frac{\partial^2 \xi}{\partial \phi_i \partial \lambda_j} + k_h^2 A^4 \frac{\partial \xi}{\partial \phi_i} \frac{\partial \xi}{\partial \lambda_j} \right] \tag{22g}$$

$$\overline{v_i u_j} = \frac{-CR^2}{A^2 f_i f_j \cos\phi_i} \psi_i Q_{ij} \left[-k_h A^2 \frac{\partial^2 \xi}{\partial \lambda_i \partial \phi_j} + k_h^2 A^4 \frac{\partial \xi}{\partial \lambda_i} \frac{\partial \xi}{\partial \phi_j} \right] \tag{22h}$$

where the partial derivates are given by

$$\frac{\partial \xi}{\partial \phi_i} = -\cos\bar{\phi}\sin\bar{\phi}\left(\lambda_i - \lambda_j\right)^2 + 2\left(\phi_i - \phi_j\right) \tag{23a}$$

$$\frac{\partial \xi}{\partial \phi_j} = -\cos\bar{\phi}\sin\bar{\phi}\left(\lambda_i - \lambda_j\right)^2 - 2\left(\phi_i - \phi_j\right) \tag{23b}$$

$$\frac{\partial \xi}{\partial \lambda_i} = 2\cos^2\bar{\phi}\left(\lambda_i - \lambda_j\right) \tag{23c}$$

$$\frac{\partial \xi}{\partial \lambda_j} = -2\cos^2\bar{\phi}\left(\lambda_i - \lambda_j\right) \tag{23d}$$

$$\frac{\partial^2 \xi}{\partial \phi_i \partial \phi_j} = \frac{1}{2}\left(\lambda_i - \lambda_j\right)^2\left(\sin^2\bar{\phi} - \cos^2\bar{\phi}\right) - 2 = -\frac{1}{2}\left(\lambda_i - \lambda_j\right)^2 \cos\left(\phi_i + \phi_j\right) - 2 \tag{23e}$$

$$\frac{\partial^2 \xi}{\partial \lambda_i \partial \lambda_j} = -2\cos^2\bar{\phi} \tag{23f}$$

$$\frac{\partial^2 \xi}{\partial \phi_i \partial \lambda_j} = 2\cos\bar{\phi}\sin\bar{\phi}\left(\lambda_i - \lambda_j\right) = \left(\lambda_i - \lambda_j\right)\sin\left(\phi_i + \phi_j\right) \tag{23g}$$

$$\frac{\partial^2 \xi}{\partial \lambda_i \partial \phi_j} = -2\cos\bar{\phi}\sin\bar{\phi}\left(\lambda_i - \lambda_j\right) = -\left(\lambda_i - \lambda_j\right)\sin\left(\phi_i + \phi_j\right) \tag{23h}$$

where $\bar{\phi} \equiv \dfrac{\phi_i + \phi_j}{2}$.

When i = j, (13), (22e), and (22f) give the variances

$$\overline{t_i^2} = C \tag{24}$$

$$\overline{u_i^2} = \overline{v_i^2} = \frac{2R^2 k_h Q_{ii} C}{f_i^2} \tag{25}$$

Substituting (13), (22a-h), (24) and (25) in (7) yields for the correlation functions corresponding to the covariances,

$$\rho_{ij}^{tt} = \exp\left[-k_h A^2 \xi\right] \Big/ \left[1 + k_p^2 \ln^2 (P_i/P_j)\right] \tag{26a}$$

$$\rho_{ij}^{ut} = -A\left(\frac{k_h}{2Q_{ii}}\right)^{1/2} P_{ij}^{i} \frac{\partial \xi}{\partial \phi_i} e^{-k_h A^2 \xi} \tag{26b}$$

$$\rho_{ij}^{tu} = -A\left(\frac{k_h}{2Q_{jj}}\right)^{1/2} P_{ij}^{j} \frac{\partial \xi}{\partial \phi_j} e^{-k_h A^2 \xi} \tag{26c}$$

$$\rho_{ij}^{vt} = \frac{A}{\cos\phi_i}\left(\frac{k_h}{2Q_{ii}}\right)^{1/2} P_{ij}^{i} \frac{\partial \xi}{\partial \lambda_i} e^{-k_h A^2 \xi} \tag{26d}$$

$$\rho_{ij}^{tv} = \frac{A}{\cos\phi_j}\left(\frac{k_h}{2Q_{jj}}\right)^{1/2} P_{ij}^{j} \frac{\partial \xi}{\partial \lambda_j} e^{-k_h A^2 \xi} \tag{26e}$$

$$\rho_{ij}^{uu} = \frac{-Q_{ij}}{2(Q_{ii}Q_{jj})^{1/2}}\left[\frac{\partial^2 \xi}{\partial \phi_i \partial \phi_j} - k_h A^2 \frac{\partial \xi}{\partial \phi_i}\frac{\partial \xi}{\partial \phi_j}\right] e^{-k_h A^2 \xi} \tag{26f}$$

$$\rho_{ij}^{vv} = \frac{-Q_{ij}}{2(Q_{ii}Q_{jj})^{1/2}\cos\phi_i\cos\phi_j}\left[\frac{\partial^2\xi}{\partial\lambda_i\partial\lambda_j} - k_h A^2 \frac{\partial\xi}{\partial\lambda_i}\frac{\partial\xi}{\partial\lambda_j}\right]e^{-k_h A^2 \xi} \tag{26g}$$

$$\rho_{ij}^{uv} = \frac{Q_{ij}}{2(Q_{ii}Q_{jj})^{1/2}\cos\phi_j}\left[\frac{\partial^2\xi}{\partial\phi_i\partial\lambda_j} - k_h A^2 \frac{\partial\xi}{\partial\phi_i}\frac{\partial\xi}{\partial\lambda_j}\right]e^{-k_h A^2 \xi} \tag{26h}$$

$$\rho_{ij}^{vu} = \frac{Q_{ij}}{2(Q_{ii}Q_{jj})^{1/2}\cos\phi_i}\left[\frac{\partial^2\xi}{\partial\lambda_i\partial\phi_j} - k_h A^2 \frac{\partial\xi}{\partial\lambda_i}\frac{\partial\xi}{\partial\phi_j}\right]e^{-k_h A^2 \xi} \tag{26i}$$

The convention adopted here for the correlation functions is that the first superscript goes with the first subscript and similarly for the second. For example, ρ_{ij}^{ut} indicates the correlation of u at the (i)th location with t at the (j)th location. This is, in general, different than ρ_{ij}^{tu}, the correlation of t at the (i)th location with u at the (j)th location. For autocorrelations like ρ_{ij}^{tt}, there is no difference.

These correlation functions can be reexpressed as the product of a "horizontal" correlation function μ and a pressure correlation function ν. The horizontal functions are

$$\mu_{ij}^{tt} = e^{-k_h A^2 \xi} \tag{27a}$$

$$\mu_{ij}^{ut} = -A\left(\frac{k_h}{2}\right)^{1/2}\frac{\partial\xi}{\partial\phi_i}\mu_{ij}^{tt} \tag{27b}$$

$$\mu_{ij}^{tu} = -A\left(\frac{k_h}{2}\right)^{1/2} \frac{\partial \xi}{\partial \phi_j} \mu_{ij}^{tt} \tag{27c}$$

$$\mu_{ij}^{vt} = \frac{A}{\cos\phi_i}\left(\frac{k_h}{2}\right)^{1/2} \frac{\partial \xi}{\partial \lambda_i} \mu_{ij}^{tt} \tag{27d}$$

$$\mu_{ij}^{tv} = \frac{A}{\cos\phi_j}\left(\frac{k_h}{2}\right)^{1/2} \frac{\partial \xi}{\partial \lambda_j} \mu_{ij}^{tt} \tag{27e}$$

$$\mu_{ij}^{uu} = -\frac{1}{2}\left[\frac{\partial^2 \xi}{\partial \phi_i \partial \phi_j} - k_h A^2 \frac{\partial \xi}{\partial \phi_i} \frac{\partial \xi}{\partial \phi_j}\right] \mu_{ij}^{tt} \tag{27f}$$

$$\mu_{ij}^{vv} = -\frac{1}{2\cos\phi_i \cos\phi_j}\left[\frac{\partial^2 \xi}{\partial \lambda_i \partial \lambda_j} - k_h A^2 \frac{\partial \xi}{\partial \lambda_i} \frac{\partial \xi}{\partial \lambda_j}\right] \mu_{ij}^{tt} \tag{27g}$$

$$\mu_{ij}^{uv} = \frac{1}{2\cos\phi_j}\left[\frac{\partial^2 \xi}{\partial \phi_i \partial \lambda_j} - k_h A^2 \frac{\partial \xi}{\partial \phi_i} \frac{\partial \xi}{\partial \lambda_j}\right] \tag{27h}$$

$$\mu_{ij}^{vu} = \frac{1}{2\cos\phi_i}\left[\frac{\partial^2 \xi}{\partial \lambda_i \partial \phi_j} - k_h A^2 \frac{\partial \xi}{\partial \lambda_i} \frac{\partial \xi}{\partial \phi_j}\right] \tag{27i}$$

The pressure correlation functions are

$$\nu_{ij}^{tt} = \frac{1}{1 + k_p^2 \ln^2 (p_i/p_j)} \tag{28a}$$

$$\nu_{ij}^{ut} = \nu_{ij}^{vt} = -P_{ij}^{i} \Big/ (Q_{ii})^{1/2} \tag{28b}$$

$$\nu_{ij}^{tu} = \nu_{ij}^{tv} = -P_{ij}^{j} \Big/ (Q_{jj})^{1/2} \tag{28c}$$

$$\nu_{ij}^{uu} = \nu_{ij}^{vv} = \nu_{ij}^{uv} = \nu_{ij}^{vu} = \frac{Q_{ij}}{(Q_{ii}\, Q_{jj})^{1/2}} \tag{28d}$$

where the values of P_{ij}^{i}, P_{ij}^{j}, and Q_{ij} are to be supplied by (17a, b, c).

b. <u>Polar Latitudes</u> ($|\phi| > 70^{\circ}$)

For polar latitudes, the horizontal correlation coefficients must be calculated using a different approximation. In place of (19a, b), let

$$x_i - x_j = A\left(\theta_i \cos\lambda_i - \theta_j \cos\lambda_j\right) \tag{29a}$$

$$y_i - y_j = A\left(\theta_i \sin\lambda_i - \theta_j \sin\lambda_j\right) \tag{29b}$$

where

$$\theta \equiv \frac{\pi}{2} - \phi \qquad \text{if} \qquad \phi > 70^{\circ} \tag{30a}$$

$$\theta \equiv \frac{\pi}{2} + \phi \qquad \text{if} \qquad \phi < -70^{\circ} \tag{30b}$$

is the co-latitude. The corresponding expression for ψ (instead of (20) and (21)) is

$$\psi_2(\lambda_i, \theta_i ; \lambda_j, \theta_j) = \exp(-k_h A^2 \eta) \tag{31}$$

where $$\eta \equiv (\theta_i \cos\lambda_i - \theta_j \cos\lambda_j)^2 + (\theta_i \sin\lambda_i - \theta_j \sin\lambda_j)^2 \tag{32}$$

Proceeding as before, but in polar coordinates instead of spherical coordinates, results in the following high-latitude forms for the horizontal correlation functions:

$$\mu_{ij}^{tt} = e^{-k_h A^2 \eta} \tag{33a}$$

$$\mu_{ij}^{ut} = -A\left(\frac{k_h}{2}\right)^{1/2} \frac{\partial \eta}{\partial \theta_i} \mu_{ij}^{tt} \tag{33b}$$

$$\mu_{ij}^{tu} = -A\left(\frac{k_h}{2}\right)^{1/2} \frac{\partial \eta}{\partial \theta_j} \mu_{ij}^{tt} \tag{33c}$$

$$\mu_{ij}^{vt} = \frac{A}{\theta_i}\left(\frac{k_h}{2}\right)^{1/2} \frac{\partial \eta}{\partial \lambda_i} \mu_{ij}^{tt} \tag{33d}$$

$$\mu_{ij}^{tv} = \frac{A}{\theta_j}\left(\frac{k_h}{2}\right)^{1/2} \frac{\partial \eta}{\partial \lambda_j} \mu_{ij}^{tt} \tag{33e}$$

$$\mu_{ij}^{uu} = -\frac{1}{2}\left[\frac{\partial^2 \eta}{\partial \theta_i \partial \theta_j} - k_h A^2 \frac{\partial \eta}{\partial \theta_i} \frac{\partial \eta}{\partial \theta_j}\right] \mu_{ij}^{tt} \tag{33f}$$

$$\mu_{ij}^{vv} = -\frac{1}{2\theta_i \theta_j}\left[\frac{\partial^2 \eta}{\partial \lambda_i \partial \lambda_j} - k_h A^2 \frac{\partial \eta}{\partial \lambda_i} \frac{\partial \eta}{\partial \lambda_j}\right] \mu_{ij}^{tt} \tag{33g}$$

$$\mu_{ij}^{uv} = \frac{1}{2\theta_j}\left[\frac{\partial^2 \eta}{\partial\theta_i \partial\lambda_j} - k_h A^2 \frac{\partial \eta}{\partial\theta_i}\frac{\partial \eta}{\partial\lambda_j}\right]\mu_{ij}^{tt} \tag{33h}$$

$$\mu_{ij}^{vu} = \frac{1}{2\theta_i}\left[\frac{\partial^2 \eta}{\partial\lambda_i \partial\theta_j} - k_h A^2 \frac{\partial \eta}{\partial\lambda_i}\frac{\partial \eta}{\partial\theta_j}\right]\mu_{ij}^{tt} \tag{33i}$$

The pressure correlation functions (28) remain the same.

III. Results

In the following discussion, it must be kept in mind that we are dealing with correlations between the deviations of temperatures and wind components from "first-guess" values (forecast or climatology or a blend), and not with correlations between raw temperature and wind components themselves. The various assumptions implicit in these correlations are more acceptable when deviational quantities are used in the analysis.

Examples of the horizontal correlation functions μ_{ij}^{tt}, μ_{ij}^{ut}, etc., are shown in Figures 1 through 8. The (i)th observation location (or analysis point location) is indicated by the circled point at the center of each figure and is at 30°N. The figures show the values the correlation functions have for (j)th observation locations in the vicinity of the (i)th point. For example, when the (j)th point is located 5° north and 5° east of the (i)th point, Figure 1 indicates that μ_{ij}^{tt} = 0.6, Figure 2 indicates that μ_{ij}^{ut} = -0.47, etc. A value of $k_h = 10^{-6}$ km^{-2} is used throughout.

It may be noted that the deviational temperature auto-correlation function μ_{ij}^{tt} is isotropic. The rest of the horizontal correlation functions, all derived from μ_{ij}^{tt}, are anisotropic. These horizontal functions are essentially the same as those of Rutherford (1973) and Schlatter (1975), except that theirs are for correlations between heights and wind components.

Figures 1 through 8 may be compared with Figure 9 reproduced from Buell (1959), in which correlations between heights and wind components on an isobaric surface are computed from observed-minus-climatology differences for real data. The general agreement, after normalization, with the "horizontal" (actually constant pressure) correlations of Figures 1 through 8 is apparent. This indicates that the geostrophic relationship, hence also the thermal wind relationship, is a valid constraint for the correlations involving wind components in the latitudes of the United States.

Figures 10 through 12 show the shapes of the pressure correlation functions as given by equations (28) for the (i)th point at 800, 500, and 200 mb, respectively. A value of $k_p = 5$ is assumed in computing these profiles. The ν_{ij}^{tt} curves agree roughly with vertical temperature correlations obtained by Rutherford (unpublished manuscript, 1974). The ν_{ij}^{ut} curves indicate that the wind at a given level is well correlated with the temperatures below that level but poorly correlated with temperatures at higher levels. Conversely, the ν_{ij}^{tu} curves show the temperatures to be highly correlated with winds above but poorly correlated with winds below. These are the expected results when the thermal wind relationship is used.

The correlation curves ν_{ij}^{uu} (identically the same as ν_{ij}^{vv} , ν_{ij}^{uv}, and ν_{ij}^{vu}) for the wind components show maximum correlation with other wind components at the same level, as expected. A surprising feature of these curves is their "broadness"; the winds are indicated to correlate fairly well at all levels for the assumed temperature autocorrelation profile, which is quite narrow in these examples. This is presumably a result of the integrated effect of the thermal wind, which interrelates the winds at all levels in the atmosphere. If the terms $\overline{u_{io} u_{jo}}$, etc., of equations (12e-h) are carried along in the theoretical development rather than being neglected, their effect would be to act in the direction of making the ν_{ij}^{uu} curves still broader and the interrelatedness of winds at all levels still greater. Since the curves are likely to be too broad already (because of a too literal application of the thermal wind constraint?), it seems judicious to neglect the surface correlation terms until calculations of vertical correlations can be computed from rawinsonde data and compared with the correlations of Figures 10-12.

The actual choices of k_h and k_p will have to be based on calculations of deviational temperature autocorrelations for real data. It may be necessary or desirable to use different values of these coefficients at different latitudes or in different regions. It may also prove to be better to use an empirically determined function for ν_{ij}^{uu} rather than one derived from ν_{ij}^{tt} through the thermal wind relation.

IV. Modification for the Tropics

In equatorial latitudes, the thermal wind relation cannot be used to relate temperatures and winds, thus the multivariate analysis must be gradually decoupled as the equator is approached. Probably the best way to do this is to replace equation (2) with

$$T_g \approx \tilde{T}_g + \sum_{i=1}^{l} a_i t_i + F(\phi)\left\{\sum_{j=1}^{m} b_j u_j + \sum_{k=1}^{n} c_k v_k\right\} \tag{34}$$

where $F(\phi)$ is a function of latitude which is near unity at the higher latitudes but decreases to zero at the equator (or perhaps at $\pm 10^{o}$, say). Similarly, if the winds are analyzed multivariately at the higher latitudes,

$$U_g \approx \tilde{U}_g + F(\phi)\sum_{i=1}^{l} a_i t_i + \sum_{j=1}^{m} b_j u_j + \sum_{k=1}^{n} c_k v_k \tag{35}$$

and similarly for V_g. Actually, the inclusion of temperature data in the wind analysis near the equator would cause far more serious errors than would the inclusion of wind data in the temperature analysis. In the latter case, the thermal wind relation would force these winds to have very low weights in determining the analyzed temperature anyway, unless the wind speeds were extremely high.

The actual choice of $F(\phi)$ will have to be determined experimentally on the basis of real data analyses. If only the temperatures, and not the winds, are analyzed multivariately, it may not be necessary to use such a function for decoupling since the winds would have minimal influence on the analyzed temperatures near the equator anyway.

V. Modification for Errors in Observations

The discussion thus far has assumed observations with no errors. In practice, it is important to allow for the observational errors in a systematic way. Other factors being equal, an observation with a large probable error should obviously receive a smaller weight than an observation with a small probable error. If the error of each observation were known exactly, it would be a simple matter to "correct" the observations, but, for most meteorological observations, only an estimate of the statistical error variance is in fact known. Thus we know that the standard deviation of radiosonde temperature errors is approximately $1^{\circ}C$, whereas that of satellite VTPR temperature errors is approximately twice as great in the troposphere.

In order to show how these differing errors can be accounted for in the analysis scheme, let us return to equations (5) and reconsider the meaning of the covariances there when observational error is included. Let $\overline{\hat{f}_i\hat{f}_j}$ be a typical autocovariance, where $\hat{f}_i$ and $\hat{f}_j$ are either both deviational temperature observations or deviational wind component observations, and the hats indicate that the observations are to some degree erroneous. Then

$$\overline{\hat{f}_i\hat{f}_j} = \overline{(f_i+\varepsilon_i)(f_j+\varepsilon_j)} = \overline{f_if_j} + \overline{f_i\varepsilon_j} + \overline{f_j\varepsilon_i} + \overline{\varepsilon_i\varepsilon_j} \tag{36}$$

where ε_i and ε_j are the statistical error standard deviations of observations f_i and f_j. The first term, $\overline{f_if_j}$, on the right-hand side of (36) is the "true" covariance of observations f_i and f_j which we have been concerned with previously.

For most kinds of observations, the observational error at one location is considered to be independent of the observational error at another location. Also, the errors are not considered to be correlated in any way with the "true" values. Then (36) reduces to

$$\overline{\hat{f}_i^2} = \overline{f_i^2} + \overline{\varepsilon_i^2}, \qquad i = j \tag{37a}$$

$$\overline{\hat{f}_i\hat{f}_j} = \overline{f_if_j}, \qquad i \neq j \tag{37b}$$

Here $\overline{\hat{f}_i^2}$ is the "inflated" variance, $\overline{f_i^2}$ the "true" variance, and $\overline{\varepsilon_i^2}$ the error variance of observation f_i. This result likely applies to all conventional types of observations and probably also to visually determined satellite winds.

The exceptions to the above are satellite-derived temperature observations. There are qualitative evidence and physical reasons for believing that the errors of these temperature observations are spatially correlated along an orbital pass. Thus, for satellite temperatures, (36) becomes

$$\overline{\hat{f}_i \hat{f}_j} = \overline{f_i f_j} + \overline{\varepsilon_i \varepsilon_j} \tag{38}$$

for all i and j where both observations are satellite temperatures. Where only one observation is a satellite temperature and the other is a conventional temperature, the errors are uncorrelated and (37a, b) apply.

Actually, there is reason to believe that the covariances $\overline{f_i \varepsilon_j}$ are not identically zero when j refers to a satellite temperature observation, but in fact are small negative quantities. This is because the temperatures as measured by satellite consistently tend to give a smoother analysis, with less detail and less pronounced extrema, as compared to "truth." See Bergman and Bonner (1974) for observational evidence of this. However, aside from the difficulty of correctly evaluating the $\overline{f_i \varepsilon_j}$ covariance, the version of "truth" appropriate for a global numerical model is in fact a smoothed version of reality. Since the magnitude of the $\overline{f_i \varepsilon_j}$ covariance is in all probability small compared to the satellite spatial error covariance $\overline{\varepsilon_i \varepsilon_j}$, there are at present no plans to consider it in the treatment of the errors.

Thus far we have discussed the autocovariances of equations (5) only. No inflation of the cross-covariances $\overline{t_i u_j}$, etc., due to observational errors occurs if the reasonable assumption is made that errors of temperature and wind observations of any type are uncorrelated with each other. The same is true of the cross-covariances $\overline{u_i v_j}$ and $\overline{v_i u_j}$ provided that the errors in the wind observations are strictly random in nature with, for example, no tendency to overestimate the u-component while simultaneously underestimating the v-component.

The additional error terms in the covariances of (37) and (38) may be normalized in the same way as the "true" covariances themselves by means of definitions (7). Thus

$$(\sigma_i^{t})^2 \equiv \frac{\overline{(\varepsilon_i^{t})^2}}{\overline{t_i^2}}, \quad (\sigma_i^{u})^2 \equiv \frac{\overline{(\varepsilon_i^{u})^2}}{\overline{u_i^2}}, \quad (\sigma_i^{v})^2 \equiv \frac{\overline{(\varepsilon_i^{v})^2}}{\overline{v_i^2}} \tag{39}$$

are added to the correlation coefficients ρ_{ii}^{tt}, ρ_{ii}^{uu}, and ρ_{ii}^{vv} (where i = j). These normalized error variances inflate the magnitude of the diagonal terms in the matrix of correlation coefficients used to solve equations (6) for the a', b', and c' weights.

Additionally, off-diagonal correlations between satellite temperature measurements will be inflated by the amounts

$$r_{ij}^{tt}\,\sigma_i^{t}\,\sigma_j^{t} \equiv \frac{\overline{\varepsilon_i^{t}\,\varepsilon_j^{t}}}{\sqrt{\overline{t_i^2}\;\overline{t_j^2}}} \tag{40}$$

where σ_i^{t} and σ_j^{t} are the standard deviations of the satellite observational errors (probably identical in most cases), and $r_{ij}^{tt}(x_i, y_i, p_i;\ x_j, y_j, p_j)$ is a functional correlation coefficient whose magnitude depends upon the separation of the two observations. The functional form of this correlation will have to be determined from satellite observational data. As a first trial, it is planned to set

$$r_{ij}^{tt} = \rho_{ij}^{tt} \tag{41}$$

i.e., the correlation of the satellite temperature errors is assumed to be the same as the correlation of the "true" deviational temperatures themselves.

The inclusion of these error terms in the system of equations (6) will have the effect of reducing the weight given to an observation in proportion to the magnitude of observational error variance, as shown by Alaka and Elvander (1972). Additionally, the weight of satellite temperature observations whose errors are highly correlated spatially (have an r_{ij}^{tt} approaching unity) will be correspondingly reduced. An indication of the effect of this spatial correlation in reducing the information value of satellite temperature observations in temperature analysis has been shown by Bergman (1974).

VI. The Guess Field

It is recommended that the values of meteorological parameters forecast by the dynamic model at grid points be blended with climatological values according to the weighted formula

$$\tilde{T} = \frac{\sigma_c^2 T_f + \sigma_f^2 T_c}{\sigma_f^2 + \sigma_c^2} \qquad (42)$$

where T_f is the forecast temperature, T_c is the climatological mean temperature (perhaps stratified by season), σ_f^2 is the forecast error variance, and σ_c^2 is the climatological variance. The resulting guess value $\tilde{T}$ would be an improvement over the forecast in the tropical regions where $\sigma_f^2 \approx \sigma_c^2$, at least in a statistical sense, although highly unusual meteorological situations might be adversely affected by this procedure. In any event, the guess field would be biased conservatively towards climatology and would still be subject to correction by the available observations in the analysis procedure.

VII. Bogus Data

No mention has been made thus far of manually entered "observations" based on subjective synoptic reasoning. As far as the analysis scheme is concerned, bogus data would be treated like any other data. A suitable standard error would have to be assigned the bogus data. This will likely be determined on the basis of experimentation with different assigned values for the error.

VIII. Data Error Check

The details of an error checking scheme for eliminating obviously erroneous data have not yet been worked out, but an initial pass of the analysis scheme for the local observation locations with the local observations themselves removed may be used as a comparative field check on observations.

IX. Summary

A multivariate analysis scheme for the simultaneous analysis of temperature and wind fields has been developed. This analysis method uses the concept of optimum interpolation as developed by Gandin, and uses the thermal wind relation to determine the functional form of all other correlations once the temperature autocorrelation has been specified. The analysis scheme is three-dimensional and can analyze for any location in the atmosphere using observations in a neighborhood volume of space. The necessary modifications of the scheme for the tropics, for inclusion of observational errors, and for "bogus" data are considered.

Several simplifying assumptions and approximations are made in the course of the development. While these necessarily reduce the degree of optimization of the resulting analysis system, it must be remembered that an analysis system must not be so complex that it is unworkable in an operational time framework. It is anticipated, moreover, that some refinements may be added to the analysis scheme once the basic form of it is working, and that modifications of an empirical type will be introduced in order to produce improved analysis results. Multivariate analysis systems in use at NCAR and in Canada are producing high quality analyses with the same kinds of simplifying assumptions that are proposed here.

The ultimate test of the analysis scheme will be the quality of the analyses that it produces compared to analyses produced by other methods. The relative quality of these analyses will have to be decided on the basis of numerical-model forecasts which use the analysis as initial data fields.

Finally, it should be remembered that even the most ingeniously devised analysis scheme will be limited by the density and quality of observational data. The most carefully optimized analysis will still be of little help in producing good forecasts in areas where the observations are sparse or are of poor quality.

REFERENCES

Alaka, M. A., and R. C. Elvander, 1972: Optimum interpolation from observations of mixed quality. Mon. Wea. Rev., 100, 612-624.

Bergman, K. H., 1974: On the contributions to analysis errors from errors in forecasts and observations. Office Note 102, NWS, National Meteorological Center, August 1974.

Bergman, K. H., and W. D. Bonner, 1974: Attempt to evaluate the errors of satellite VTPR temperature data by a climatological method. Office Note 103, NWS, National Meteorological Center, August 1974.

Buell, C. E., 1958: The correlation between wind and height on an isobaric surface. J. Meteor., 15, 502-512.

Buell, C. E., 1959: An application of turbulence theory to the synoptic scale phenomena of the atmosphere. Report No. KND 96-95-16 (FR), Kaman Sciences Corporation, Colorado Springs, Colorado, in four volumes, Summary Report, Contract DA-04-495-ORD-1195, U. S. Army Ballistic Missile Agency.

Buell, C. E., 1972: Variability of wind with distance and time on an isobaric surface. J. Applied Meteor., 11, 1085-1091.

Gandin, L. S., 1963: Objective Analysis of Meteorological Fields. Gidrometeorologicheskoe Izdatelstvo, Leningrad. Translation by Israel Program for Scientific Translations, Jerusalem, 1965, 242 pp.

Kluge, J., 1970: Ausnutzung der Winddaten bei der numerischen analyse des geopotentials der AT der mittleren troposphäre mittels optimaler interpolation. Zeitscrift für Meteorologie, 21, 286-292.

Rutherford, I. D., 1973: Experiments on the updating of P.E. forecasts with real wind and geopotential data. Preprint, Third Conference on Probability and Statistics in Atmospheric Sciences, June 19-22, 1973, Boulder, Colorado; published by AMS, Boston, Mass., pp. 198-201.

Schlatter, T. W., 1975: Some experiments with a multivariate statistical objective analysis scheme. Mon. Wea. Rev., 103, 246-257.

APPENDIX

The following proofs of covariance derivative relationships are based on proofs provided to the author by Dr. T. W. Schlatter of NCAR.

Given $\quad f_i(x_i, y_i, p_i) \qquad f_j(x_j, y_j, p_j)$, then

$$1.)\ \left(\overline{\frac{\partial f_i}{\partial x_i} f_j}\right) = \overline{\left\{\lim_{\Delta x_i \to 0}\left[\frac{f_i(x_i+\Delta x_i, y_i, p_i) - f_i(x_i, y_i, p_i)}{\Delta x_i}\right] f_j(x_j, y_j, p_j)\right\}}$$

$$= \overline{\left\{\lim_{\Delta x_i \to 0}\left[\frac{f_i(x_i+\Delta x_i, y_i, p_i) f_j(x_j, y_j, p_j) - f_i(x_i, y_i, p_i) f_j(x_j, y_j, p_j)}{\Delta x_i}\right]\right\}}$$

$$= \lim_{\Delta x_i \to 0}\left\{\frac{\overline{f_i(x_i+\Delta x_i, y_i, p_i) f_j(x_j, y_j, p_j)} - \overline{f_i(x_i, y_i, p_i) f_j(x_j, y_j, p_j)}}{\Delta x_i}\right\}$$

$$= \frac{\partial}{\partial x_i}\left(\overline{f_i f_j}\right)$$

Similarly, $\left(\overline{\frac{\partial f_i}{\partial y_i} f_j}\right) = \frac{\partial}{\partial y_i}\left(\overline{f_i f_j}\right), \quad \left(\overline{\frac{\partial f_i}{\partial p_i} f_j}\right) = \frac{\partial}{\partial p_i}\left(\overline{f_i f_j}\right)$,

$$2.)\ \left(\overline{f_i \frac{\partial f_j}{\partial x_j}}\right) = \overline{\left\{f_i(x_i, y_i, p_i) \lim_{\Delta x_j \to 0}\left[\frac{f_j(x_j+\Delta x_j, y_j, p_j) - f_j(x_j, y_j, p_j)}{\Delta x_j}\right]\right\}}$$

$$= \overline{\left\{\lim_{\Delta x_j \to 0}\left[\frac{f_i(x_i, y_i, p_i) f_j(x_j+\Delta x_j, y_j, p_j) - f_i(x_i, y_i, p_i) f_j(x_j, y_j, p_j)}{\Delta x_j}\right]\right\}}$$

$$= \lim_{\Delta x \to 0}\left\{\frac{\overline{f_i(x_i, y_i, p_i) f_j(x_j+\Delta x, y_j, p_j)} - \overline{f_i(x_i, y_i, p_i) f_j(x_j, y_j, p_j)}}{\Delta x_j}\right\}$$

$$= \frac{\partial}{\partial x_j}\left(\overline{f_i f_j}\right)$$

Similarly, $\left(\overline{f_i \frac{\partial f_j}{\partial y_j}}\right) = \frac{\partial}{\partial y_j}\left(\overline{f_i f_j}\right), \quad \left(\overline{f_i \frac{\partial f_j}{\partial p_j}}\right) = \frac{\partial}{\partial p_j}\left(\overline{f_i f_j}\right)$

3.) $$\overline{\left(\frac{\partial f_i}{\partial x_i}\,\frac{\partial f_j}{\partial x_j}\right)} = \overline{\left\{\left[\lim_{\Delta x_i \to 0} \frac{f_i(x_i+\Delta x_i,\, y_i, P_i) - f_i(x_i, y_i, P_i)}{\Delta x_i}\right]\left[\lim_{\Delta x_j \to 0} \frac{f_j(x_j+\Delta x_j,\, y_j, P_j) - f_j(x_j, y_j, P_j)}{\Delta x_j}\right]\right\}}$$

$$= \overline{\left\{\lim_{\Delta x_i \to 0}\,\lim_{\Delta x_j \to 0}\left[\frac{f_i(x_i+\Delta x_i) f_j(x_j+\Delta x_j) - f_i(x_i+\Delta x_i) f_j(x_j) - f_i(x_i) f_j(x_j+\Delta x_j) + f_i(x_i) f_j(x_j)}{\Delta x_i\, \Delta x_j}\right]\right\}}$$

$$= \lim_{\Delta x_i \to 0}\,\lim_{\Delta x_j \to 0}\left\{\frac{\overline{f_i(x_i+\Delta x_i) f_j(x_j+\Delta x_j)} - \overline{f_i(x_i+\Delta x_i) f_j(x_j)} - \overline{f_i(x_i) f_j(x_j+\Delta x_j)} + \overline{f_i(x_i) f_j(x_j)}}{\Delta x_i\, \Delta x_j}\right\}$$

$$= \lim_{\Delta x_i \to 0} \frac{1}{\Delta x_i}\left\{\lim_{\Delta x_j \to 0}\left[\frac{\overline{f_i(x_i+\Delta x_i) f_j(x_j+\Delta x_j)} - \overline{f_i(x_i+\Delta x_i) f_j(x_j)}}{\Delta x_j}\right]\right.$$

$$\left. - \lim_{\Delta x_j \to 0}\left[\frac{\overline{f_i(x_i) f_j(x_j+\Delta x_j)} - \overline{f_i(x_i) f_j(x_j)}}{\Delta x_j}\right]\right\}$$

$$= \lim_{\Delta x_i \to 0} \frac{1}{\Delta x_i}\left\{\frac{\partial}{\partial x_j}\left[\overline{f_i(x_i+\Delta x_i) f_j(x_j)}\right] - \frac{\partial}{\partial x_j}\left[\overline{f_i(x_i) f_j(x_j)}\right]\right\}$$

$$= \frac{\partial^2}{\partial x_i\, \partial x_j}\left(\overline{f_i f_j}\right)$$

Proofs for other relations involving second-order partial derivatives are similar.

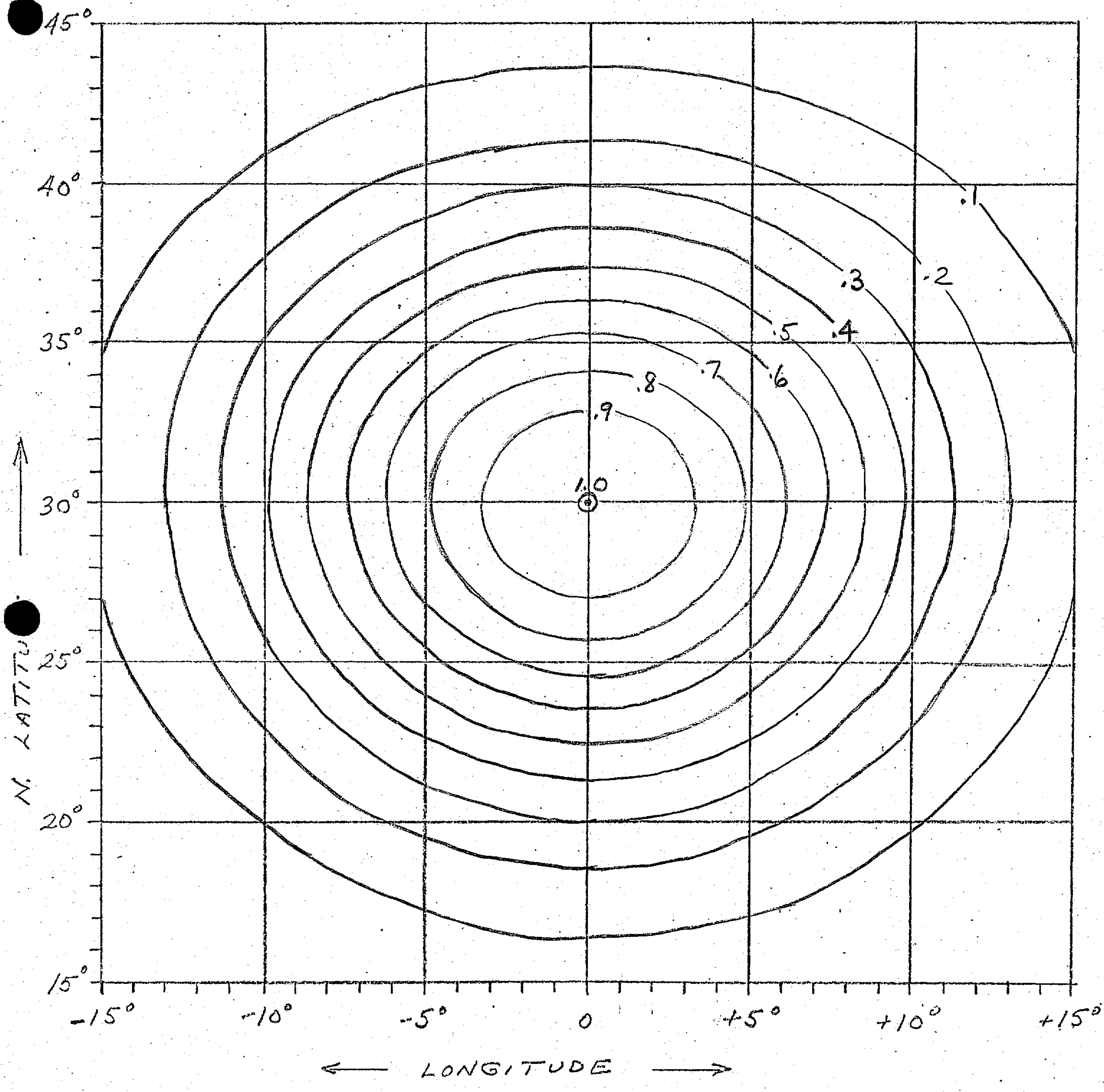

Fig. 1. Horizontal Correlation μ_{ij}^{tt} at 30°N

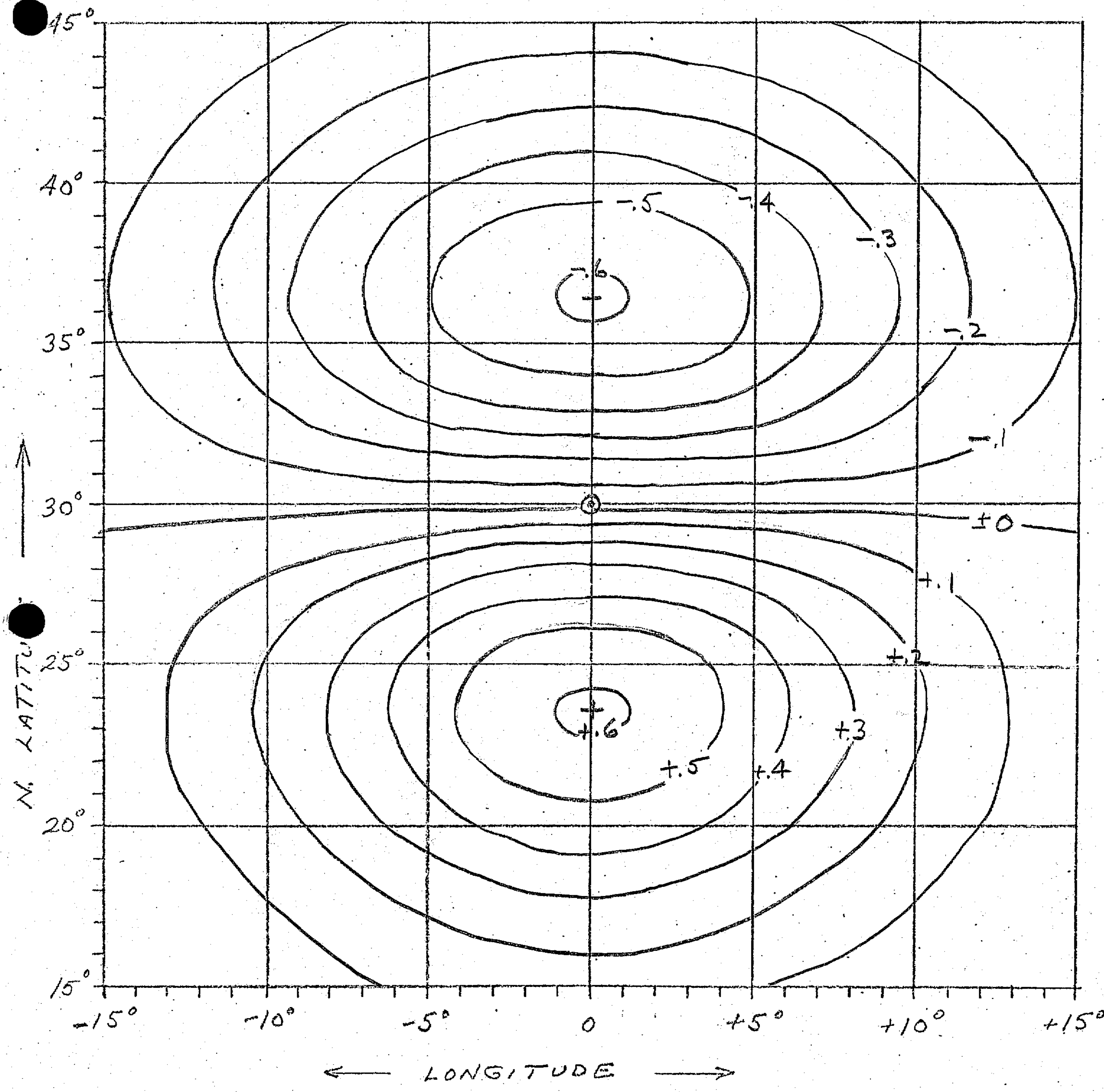

Fig. 2. Horizontal Correlation μ_{ij}^{ut} at 30°N

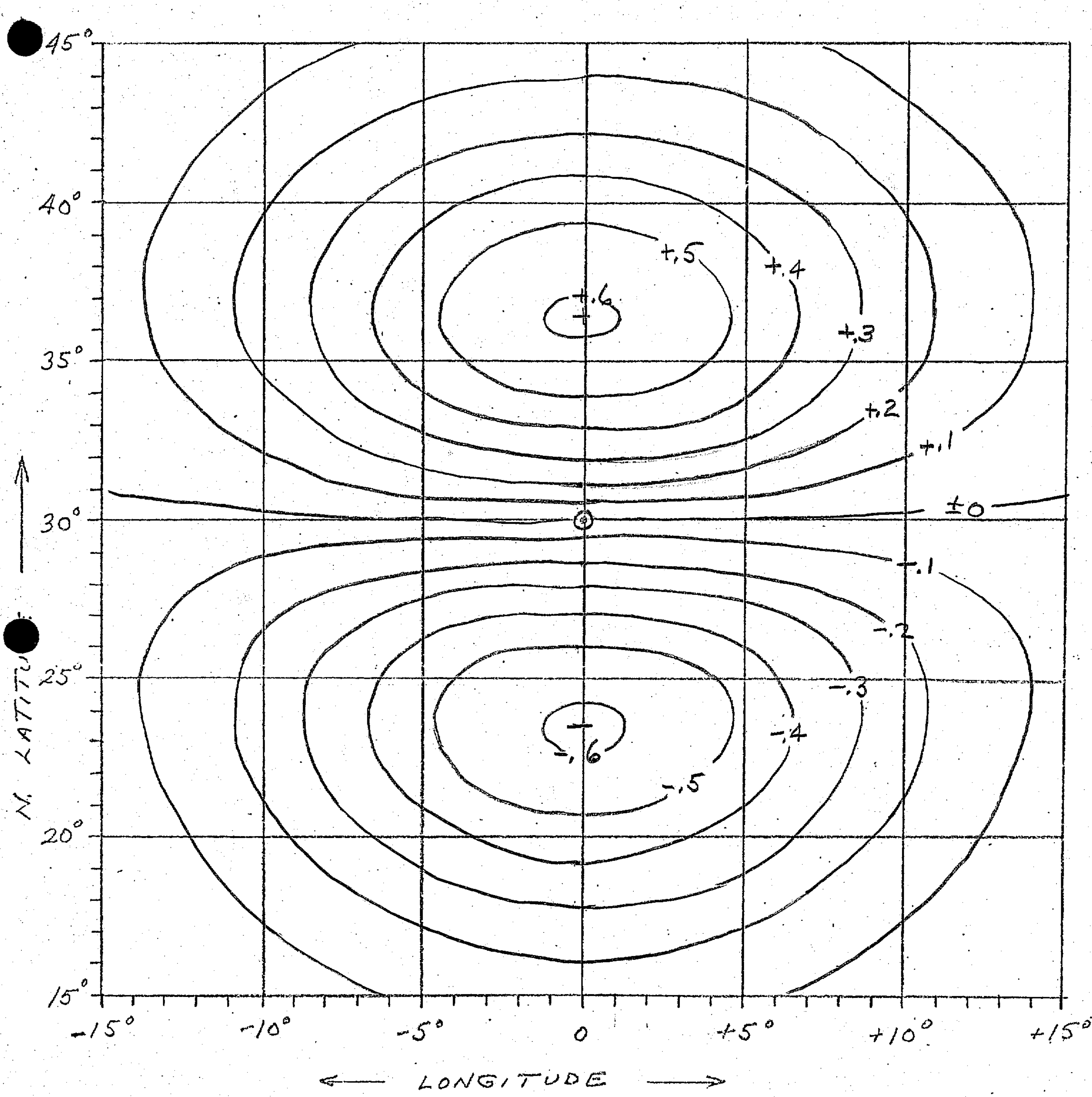

Fig. 3. Horizontal Correlation μ_{ij}^{tu} at 30°N

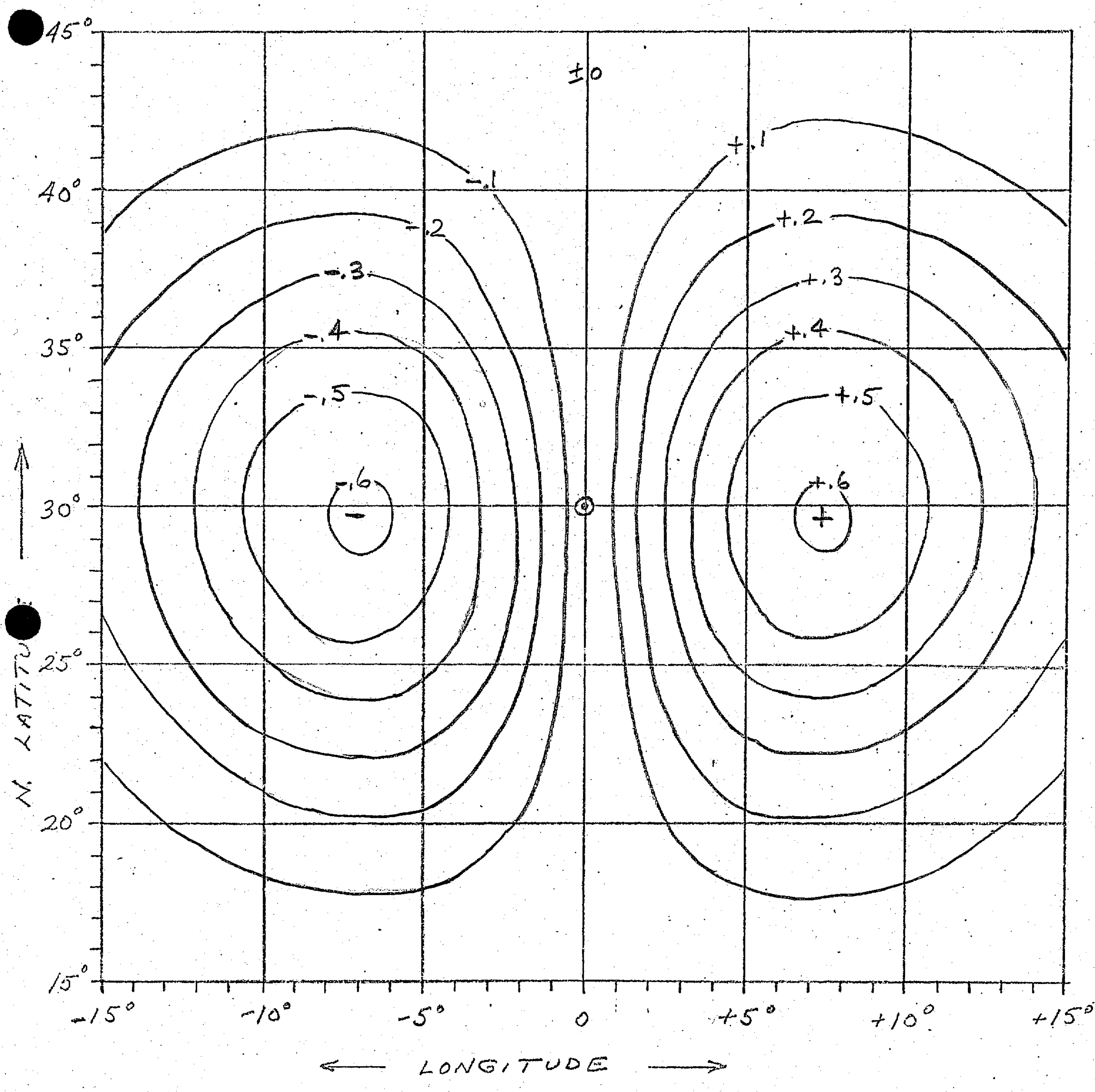

Fig. 4. Horizontal Correlation μ_{ij}^{vt} at 30°N

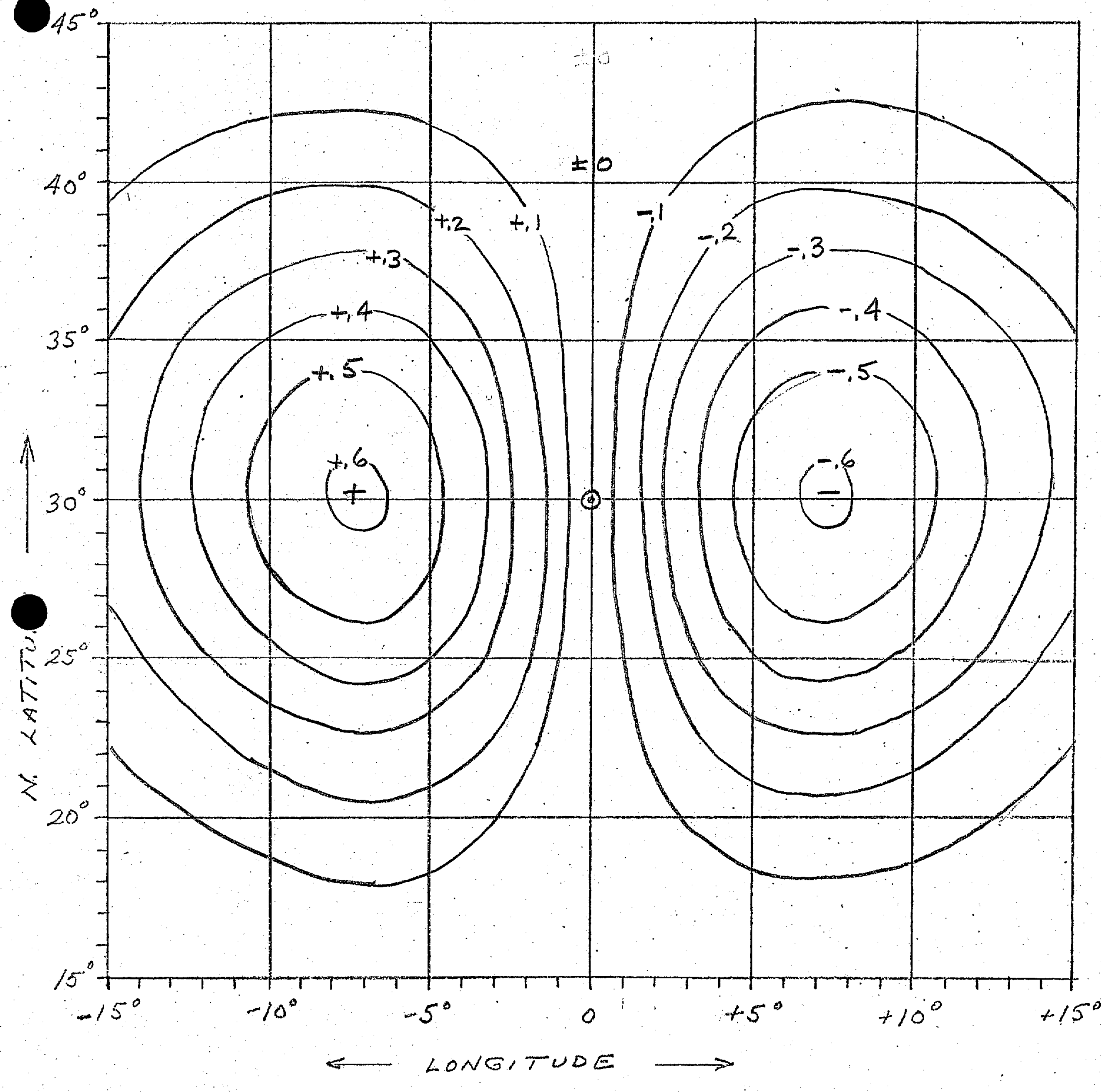

Fig. 5. Horizontal Correlation μ_{ij}^{tv} at 30°N

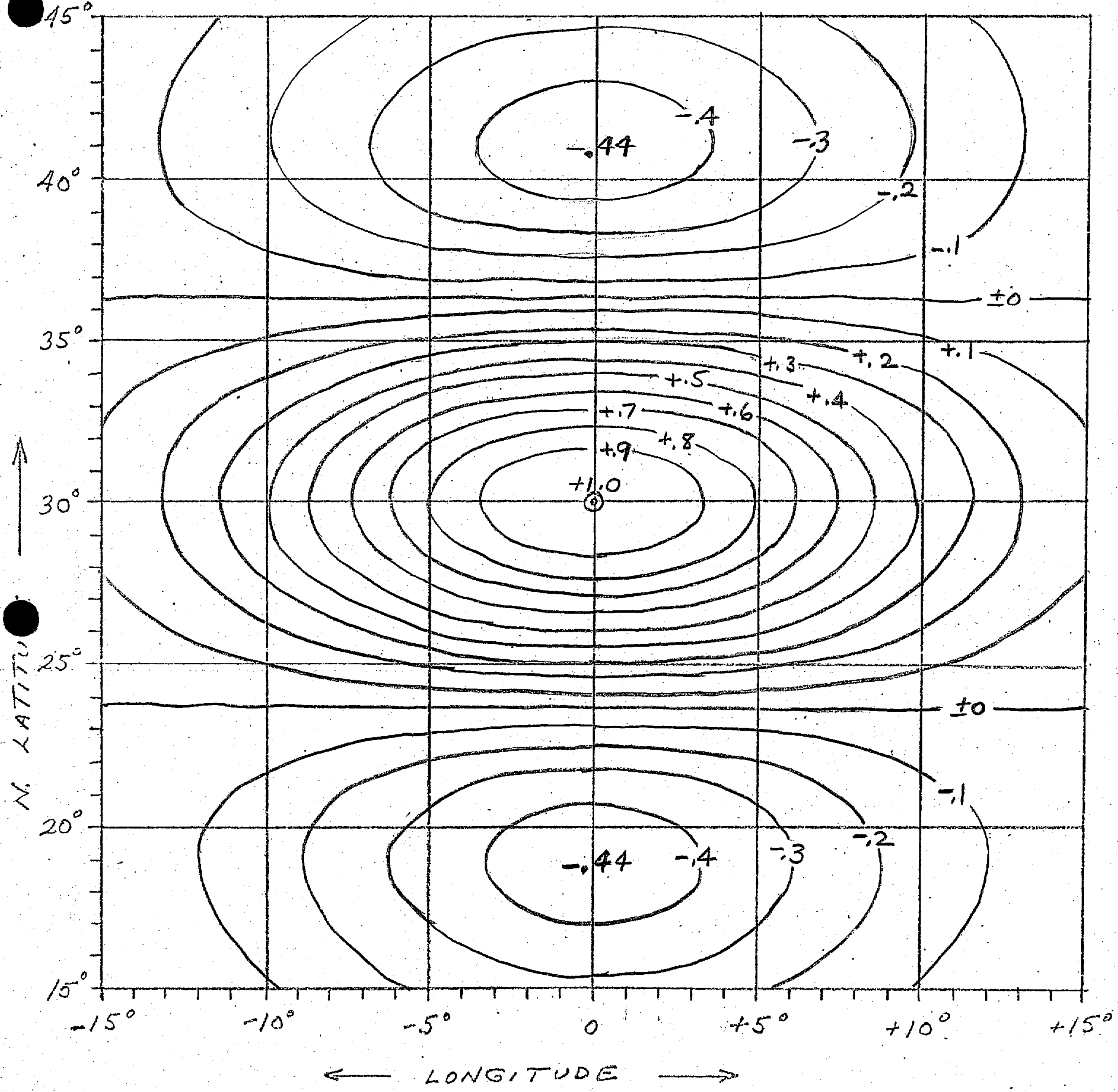

Fig. 6. Horizontal Correlation μ_{ij}^{uu} at 30°N

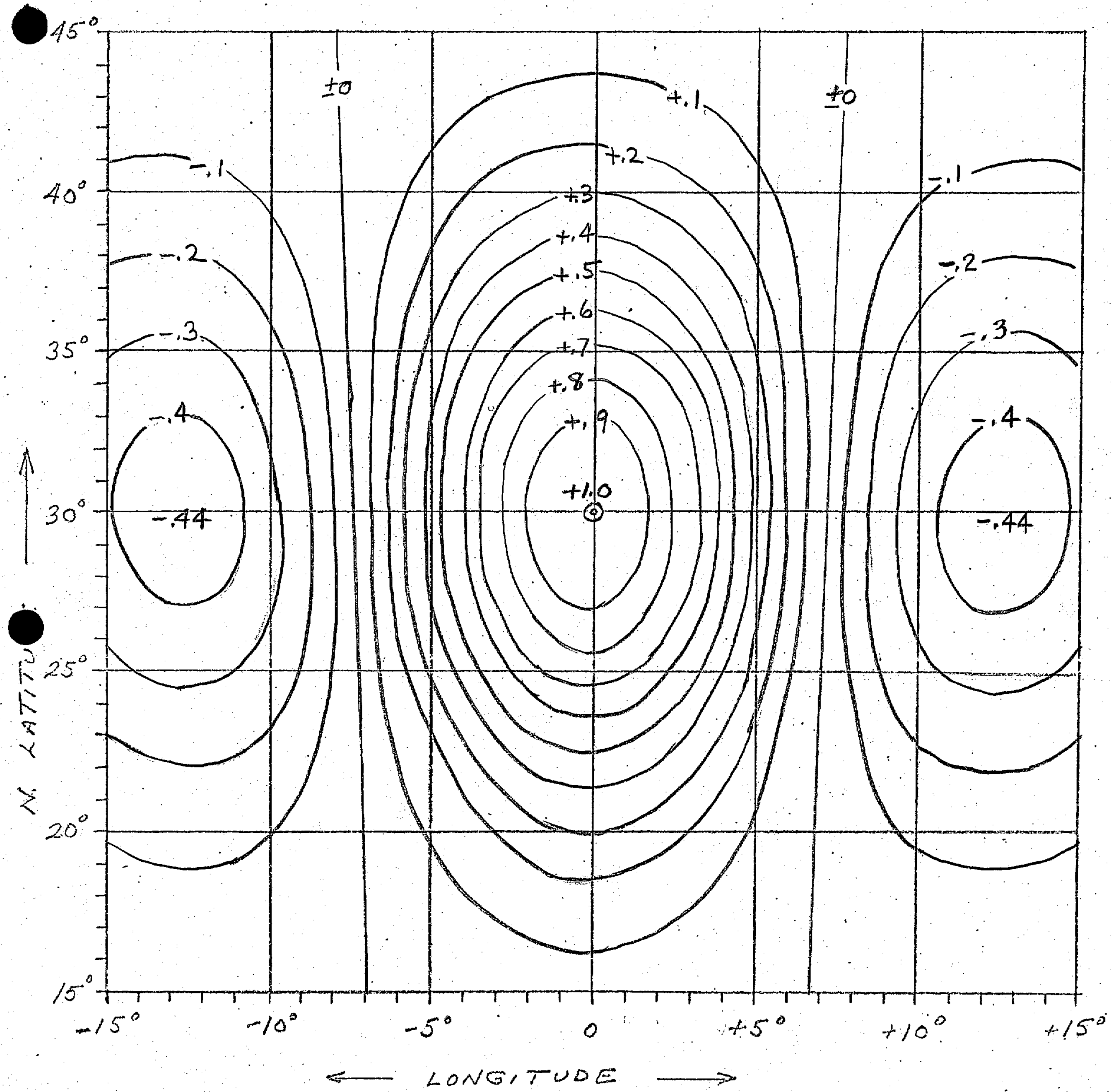

Fig. 7. Horizontal Correlation μ_{ij}^{vv} at 30°N

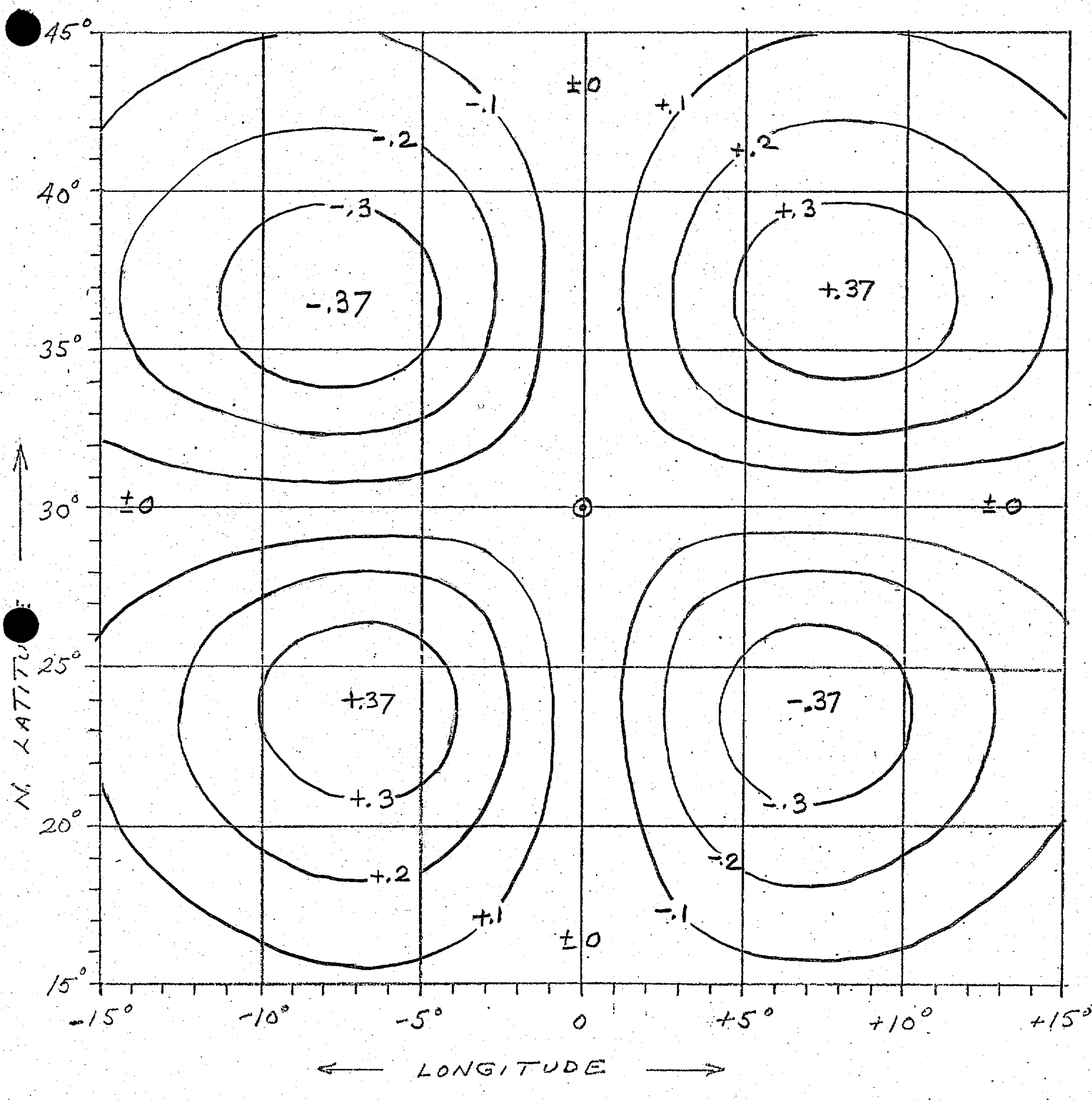

Fig. 8.
HORIZONTAL CORRELATIONS μ_{ij}^{uv}, μ_{ij}^{vu} AT 30°N

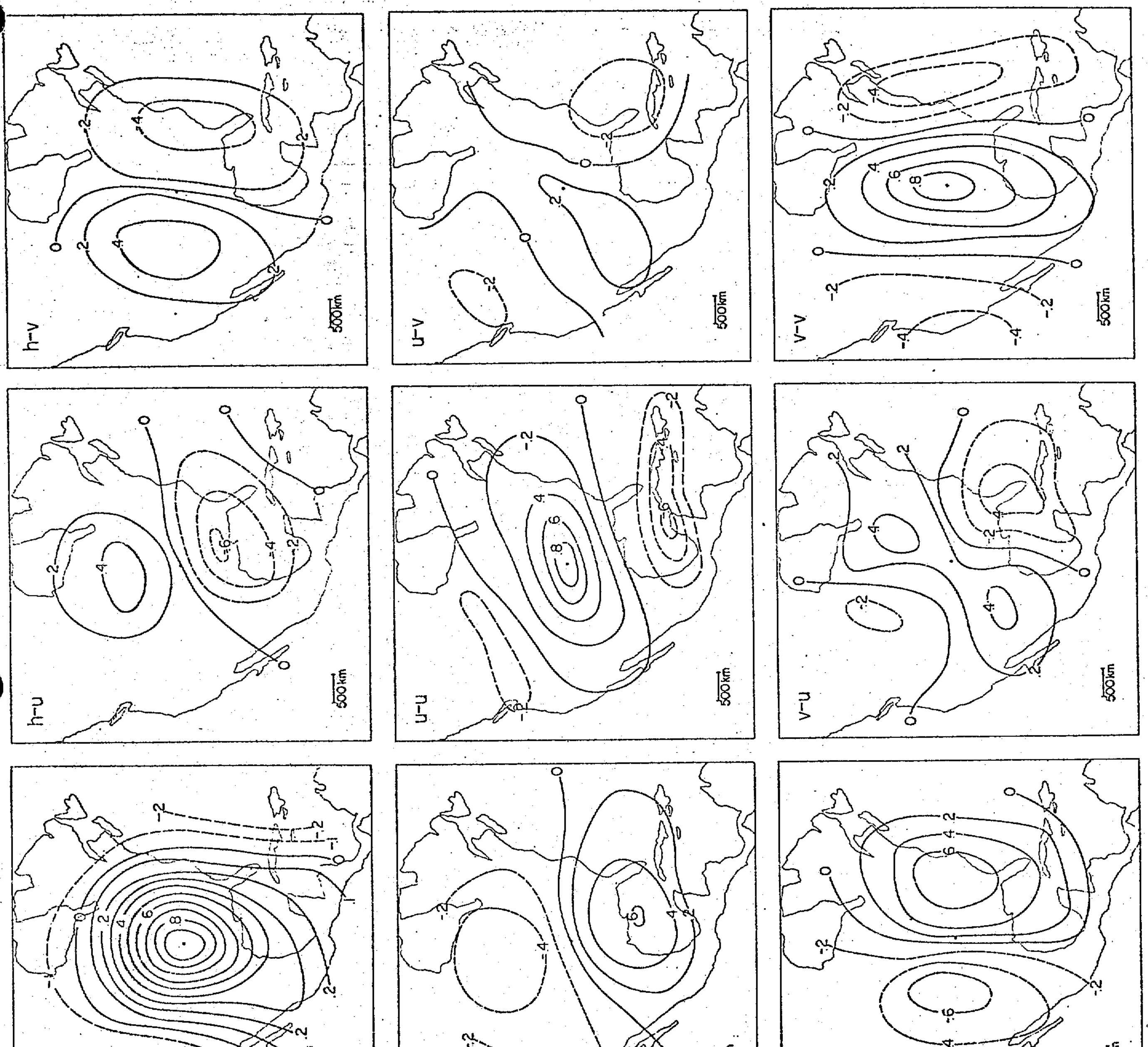

Fig. 9 Correlations among h, u, and v as calculated by Buell (1959) from 5 years of U.S. radiosonde data. Master station is Columbia, Missouri.

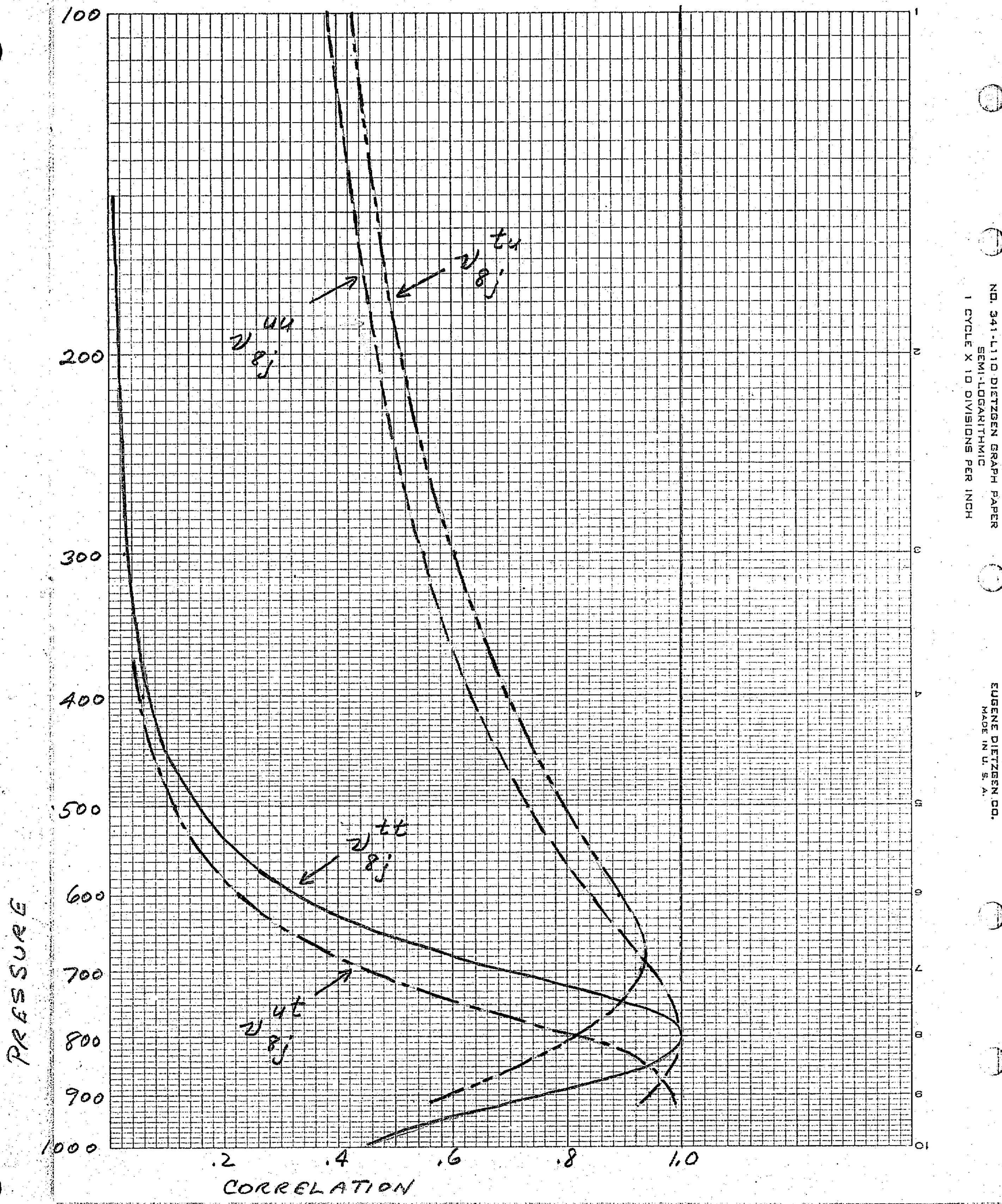

Fig. 10. Pressure Correlation Functions for (i)th Point at 800 mb.

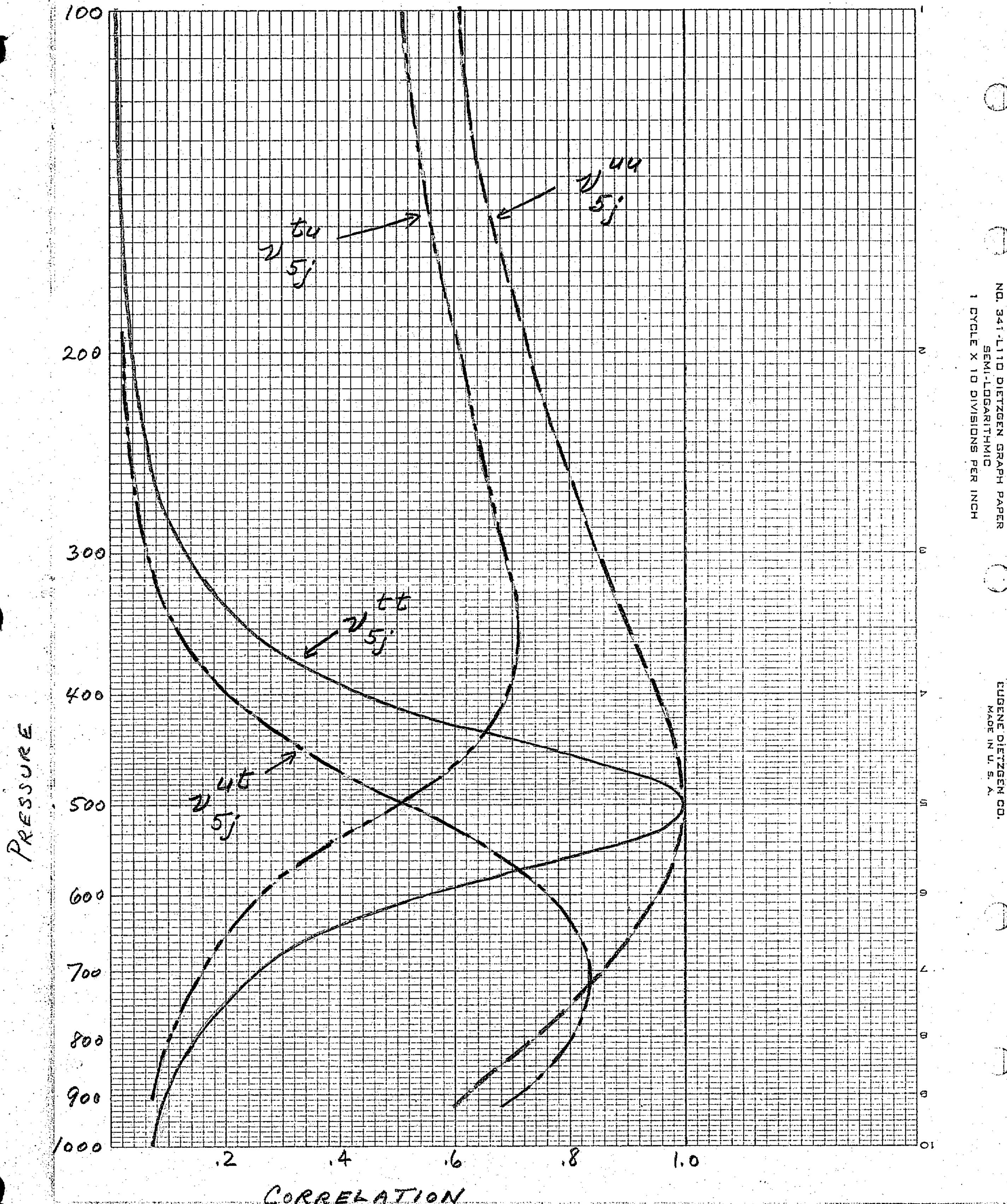

Fig. 11. Pressure Correlation Functions for (i)th Point at 500 mb.

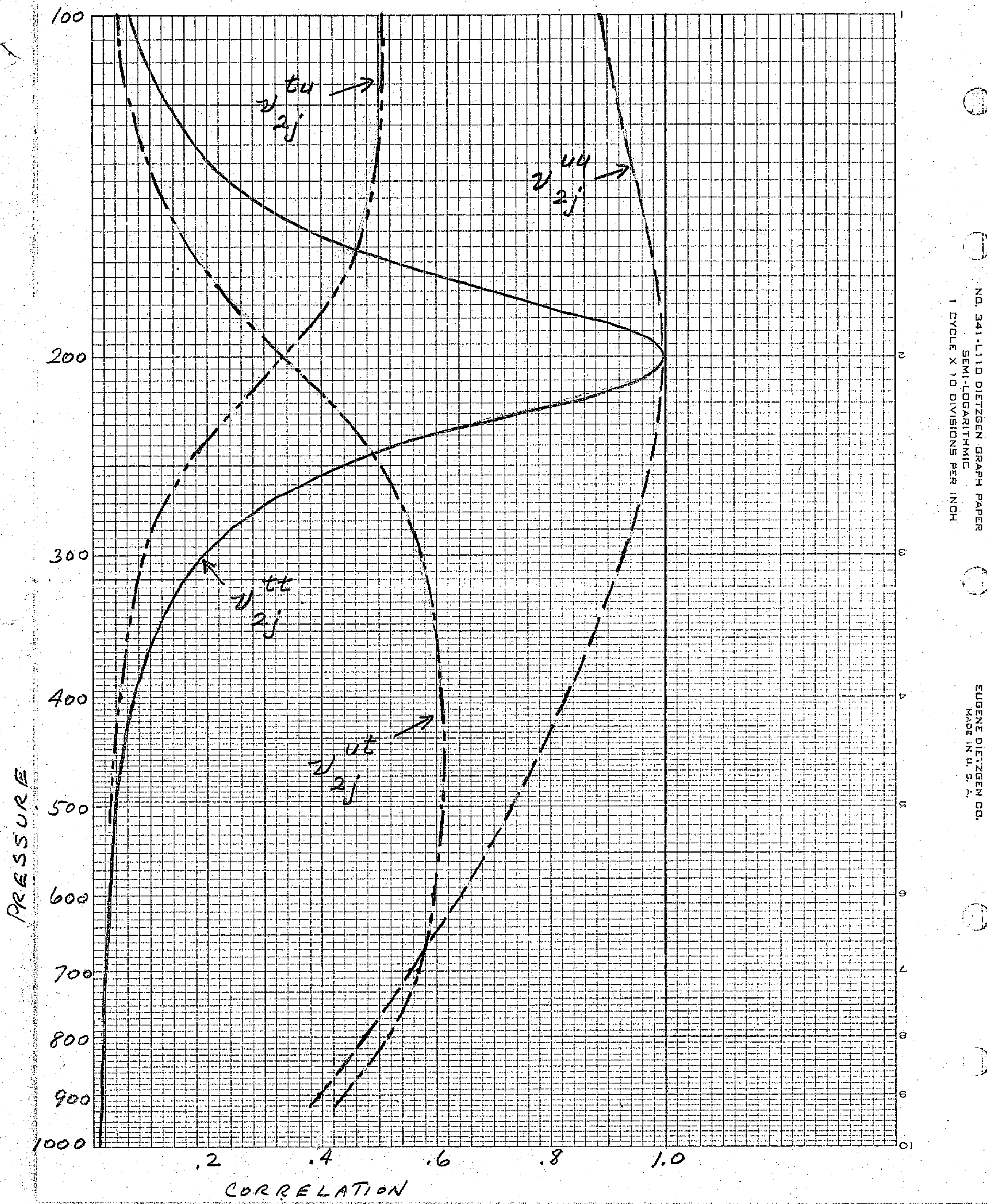

Fig. 12. Pressure Correlation Functions for (i)th Point at 200 mb

Office Note 116, "Multivariate objective analysis of temperature and wind fields," by K. H. Bergman, Development Division.

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